國立中山大學 101 學年度碩士暨碩士專班招生考試試題

科目：線性代數【通訊所碩士班甲組】

1. Given the following matrix: (12%)

$$\begin{bmatrix} 2 & 1-i \\ 1+i & 1 \end{bmatrix}$$

Determine whether it is Hermitian, unitary, singular, and positive definite. Please explain your reasons to each answer.

2. Consider the following 3×3 matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(i) Find the eigenvalue decomposition of $\mathbf{A}$ (8%)
(ii) Find a matrix $\mathbf{L}$ such that $\mathbf{L}\mathbf{L}^T = \mathbf{A}$ (5%)
(iii) Find the singular values of the matrix $\mathbf{L}$ (5%)

3. Consider three vectors :

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

(i) Apply the Gram-Schmidt process to $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, $\boldsymbol{u}_3$ to form a set of orthonormal bases. (5%)
(ii) Find the orthogonal projection of a vector $\boldsymbol{b} = [2 \quad -1 \quad 3 \quad 1 \quad 1]^T$ on the space spanned by $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$. (5%)
(iii) Find the QR decomposition of (5%)

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(iv) Find a solution of $\boldsymbol{x} = [x_1 \quad x_2 \quad x_3]^T$, such that $\|\mathbf{U}\boldsymbol{x} - \boldsymbol{b}\|^2$ is minimized. (5%)

4. (15%) Let $t: \mathcal{P}_2 \to \mathcal{P}_2$ be
$$a_0 + a_1x + a_2x^2 \to (5a_0 + 6a_1 + 2a_2) - (a_1 + 8a_2)x + (a_0 - 2a_2)x^2.$$
Find the eigenvalues and the associated eigenvectors of the map t.

5. (10%) Show that if the set $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is linearly independent set then so is the set $\{\boldsymbol{u}, \boldsymbol{u}+\boldsymbol{v}, \boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w}\}$.

6. (15%) Show that matrices of this form are not diagonalizable.
$$\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}, c \neq 0$$

7. (10%) **T** is said to be positive definite if $\langle \mathbf{T}(x), x \rangle > 0$ for all $x \neq 0$. Let **T** and **U** be positive operators on an inner product space **V**. Prove
   (i) (5%) $\mathbf{T} + \mathbf{U}$ is positive definite.
   (ii) (5%) If $c > 0$, then $c\mathbf{T}$ is positive definite.

1. (15 %) For each part, if the statement is true, please write a circle ("o"). If the statement is wrong, then mark it as ("x"). You do NOT need to provide any justification.
   (a) (5 %) (   ). Let X, Y be independent random variables, both uniformly distributed on (-1/2,1/2). Then $X+Y$ is uniformly distributed on (-1,1).
   (b) (5 %) (   ). Assume X, Y be independent random variables, both normally distributed with parameters (μ,σ^2) being $(2,3^2)$ and $(-2,4^2)$. Then $X+Y$ is normally distributed with parameters $(0,5^2)$.
   (c) (5 %) (   ). Let the joint density function of X, Y be $f(x,y)=\frac{4}{\pi}\exp\left(-(x^2+y^2)\right)$ for $x>0, y>0$, and zero otherwise. Then X and Y are independent.

2. (10 %) Assume a random variable X is uniform on $(0,L)$. Decide the probability of which new variable $\min\left(\frac{X}{L-X},\frac{L-X}{X}\right)$ is less than 1/3. (i.e., calculate $P\left(\min\left(\frac{X}{L-X},\frac{L-X}{X}\right)<\frac{1}{3}\right)$)

3. (15%) Consider two random variables X and Y with the joint distribution $f(x,y)=ce^{-(\pi x^2+4\pi y^2)}$, $-\infty<x,y<\infty$. Please decide
   (a) (5 %) c;
   (b) (5 %) $P\left(Y>0\mid X>\frac{1}{\pi}\right)$;
   (c) (5 %) $E(XY\mid Y=\pi)$.

4. (10%) A random variable X is uniform on (-2,3). If $Y=-X^2+4$, find the distribution of Y.

5. (15%) Given any two real-valued random variables X_1 and X_2 with finite second moment. Here, $\mathrm{E}\{\cdot\}$ takes the expectation with respect to X_1 and X_2. If the statement is true, please write a circle ("o"). If the statement is wrong, then mark it as ("x"). You do NOT need to provide any justification.
   (a) (5 %) (   ). $(\mathrm{E}\{X_1X_2\})^2\le\mathrm{E}\{X_1^2\}\mathrm{E}\{X_2^2\}$;
   (b) (5 %) (   ). $\mathrm{E}\{c_1X_1+c_2X_2\}\neq c_1\mathrm{E}\{X_1\}+c_2\mathrm{E}\{X_2\}$, where c_1 and c_2 are constant values;
   (c) (5 %) (   ). $\mathrm{E}\{(X_1+X_2)^2\}\le\mathrm{E}\{X_1^2\}+\mathrm{E}\{X_2^2\}$.

6. (15%) Let Y be a binomial distribution with parameters n and p; i.e., the probability distribution function of Y is given by $P(Y=y)=\binom{n}{y}p^y(1-p)^{n-y}$, $y=0,1,2,\cdots,n$. Please find
   (a) (5 %) the mean of Y,

(b) (5 %) the variance of Y,

(c) (5 %) the probability generating function of Y.

7. (10%) The joint probability density function of the random variable (X_1, X_2) is given by

$$f(x_1, x_2) = \begin{cases} c(x_1 + x_2) & 0 < x_1 < x_2 < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Are X_1 and X_2 stochastically independent? Why?

8. (10%) Let X and Y be independent normal random variables with zero mean and unit variance. Find the value of $\mathrm{E}\{X^2Y + XY^2 + X^2Y^2\}$, in which $\mathrm{E}\{\cdot\}$ takes the expectation with respect to X and Y.

1. **[20] Matched Filter:** Prove that if a signal $s(t)$ is corrupted by AWGN, the filter with an impulse response matched to $s(t)$ maximizes the output signal-to-noise ratio. The maximum SNR obtained with the matched filter is $\text{SNR}_0 = \frac{2}{N_0}\int_0^T s^2(t)\,dt = \frac{2\varepsilon}{N_0}$.

2. **[20] Hilbert Transform:** The Hilbert transform is given by $\hat{x}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(\tau)}{t-\tau}d\tau$. Prove the following properties:
   A. [5] If $x(t) = x(-t)$, then $\hat{x}(t) = -\hat{x}(-t)$.
   B. [5] If $x(t) = -x(-t)$, then $\hat{x}(t) = \hat{x}(-t)$.
   C. [5] $\int_{-\infty}^{\infty} x(t)\hat{x}(t)\,dt = 0$.
   D. [5] If $x(t) = \sin\omega_0 t$, then $\hat{x}(t) = -\cos\omega_0 t$.

3. **[20] *M*-ary PAM Modulation:** The *M*-ary PAM signals can be represented geometrically as *M* one-dimensional signal points with value $s_m = \sqrt{\frac{1}{2}\varepsilon_g}A_m$, $m = 1,2,\ldots,M$, where ε_g is the energy of the basic signal pulse $g(t)$. The amplitude values may be expressed as $A_m = (2m-1-M)d$, $m = 1,2,\ldots,M$. On the assumption of the each signal has equal probability,
   A. [5] Find the average energy.
   B. [10] Calculate the average probability of a symbol error for *M*-ary PAM.
   C. [5] Please use the result in Part A to show the probability of a symbol error for rectangular *M*-ary QAM. ($M = 2^k$, k is even)

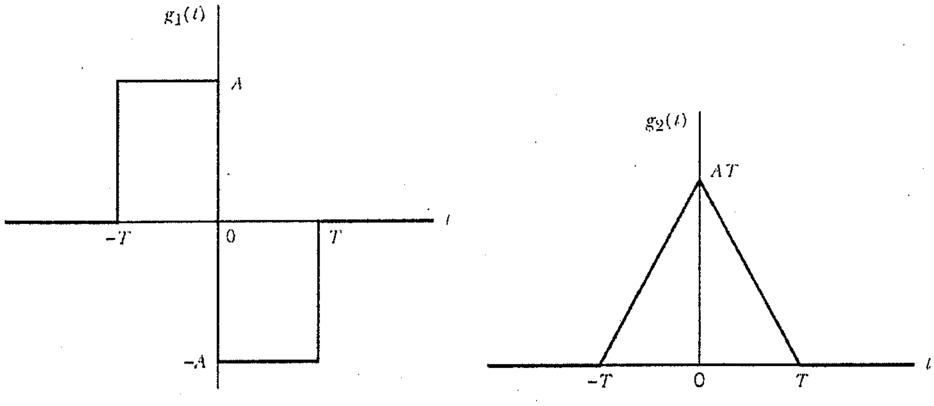
4. **[20] Band-Pass Systems**: Consider a band-pass system. The time domain received signal is $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$, where $h(t) = \text{Re}\left[\tilde{h}(t)\exp(j2\pi f_c t)\right]$ is the impulse response of a bandpass filter and $\tilde{h}(t)$ is the complex impulse response of the bandpass filter.
   A. [8] Please show that $H(f) = \frac{1}{2}\left[\tilde{H}(f-f_c) + \tilde{H}^*(-f-f_c)\right]$, where $H(f)$ and $\tilde{H}(f)$ are Fourier transform of $h(t)$ and $\tilde{h}(t)$, respectively.
   B. [12] Please show that $\tilde{y}(t) = \frac{1}{2}\tilde{h}(t) * \tilde{x}(t)$, where $\tilde{x}(t)$ and $\tilde{y}(t)$ are the complex envelope of the band-pass input and output, respectively.

5. **[20] Fourier Transform:** (Hint: You may use the attached properties of the Fourier transform.)

A. [5] Find the Fourier transform of the rectangular pulse: $g(t)=\begin{cases}1, & -\frac{1}{2}\le t\le\frac{1}{2},\\ 0, & |t|\ge\frac{1}{2}.\end{cases}$

B. [5] Find the Fourier transform of the doublet pulse $g_1(t)$ shown in Figure (a).

C. [10] Find the Fourier transform of the triangular pulse $g_2(t)$ shown in Figure (b).



(a) (b)

Properties of the Fourier Transform

Property	Mathematical Description
Linearity	$ag_1(t)+bg_2(t)\rightleftharpoons aG_1(f)+bG_2(f)$.
Time scaling	$g(at)\rightleftharpoons\frac{1}{\lvert a\rvert}G\left(\frac{f}{a}\right)$, where a is a constant.
Duality	If $g(t)\rightleftharpoons G(f)$, then $G(t)\rightleftharpoons g(-f)$.
Time shifting	$g(t-t_0)\rightleftharpoons G(f)\exp(-j2\pi ft_0)$.
Frequency shifting	$\exp(j2\pi f_c t)g(t)\rightleftharpoons G(f-f_c)$.
Area under $g(t)$	$\int_{-\infty}^{\infty}g(t)\,dt=G(0)$.
Differentiation in the time domain	$\frac{d}{dt}g(t)\rightleftharpoons j2\pi fG(f)$.
Integration in the time domain	$\int_{-\infty}^{t}g(\tau)\,d\tau\rightleftharpoons\frac{1}{j2\pi f}G(f)+\frac{G(0)}{2}\delta(f)$.
Conjugate functions	If $g(t)\rightleftharpoons G(f)$, then $g^*(t)\rightleftharpoons G^*(-f)$.
Multiplication in the time domain	$g_1(t)g_2(t)\rightleftharpoons\int_{-\infty}^{\infty}G_1(\lambda)G(f-\lambda)\,d\lambda$.
Convolution in the time domain	$\int_{-\infty}^{\infty}g_1(\tau)g_2(t-\tau)\,d\tau\rightleftharpoons G_1(f)G_2(f)$.
Rayleigh's energy theorem	$\int_{-\infty}^{\infty}\lvert g(t)\rvert^2dt=\int_{-\infty}^{\infty}\lvert G(f)\rvert^2df$.