Problem 1. (Totally, 20 points) Let X_1 and X_2 be two correlated random variables with a common state space $\{0,1\}$. It is assumed that $P\{X_1 = 0\} = \frac{11}{24}$ and $P\{X_2 = 0\} = \frac{11}{24}$. In addition, the conditional probability that $X_1 = 0$, given that $X_2 = 0$, is $\frac{2}{3}$. Furthermore, the conditional probability that $X_2 = 1$, given that $X_1 = 1$, is $\frac{9}{13}$.

(a) (5 points) Calculate the joint probability density function $f(x_1, x_2) = P\{X_1 = x_1, X_2 = x_2\}$.

(b) (5 points) Calculate the conditional probability $P\{X_1 = 0|X_2 = 1\}$.

(c) (10 points) Calculate the covariance matrix of X_1 and X_2.

Problem 2. (Totally, 35 points) Let $X_1, X_2, X_3, ...$ be a Markov chain with state space $S = \{1,2,3,4,5\}$ and transition matrix T. An infinite sequence of random variables $X_1, X_2, X_3, ...$ is said to be a Markov chain, if for each integer $n \geq 1$,

$$P\{X_{n+1} = x_{n+1}|X_n = x_n, X_{n-1} = x_{n-1}, .., X_1 = x_1\} = P\{X_{n+1} = x_{n+1}|X_n = x_n\}$$

That is, given the current state X_n, the future of a Markov chain, X_{n+1}, is independent of all past states, $X_{n-1}, X_{n-2}, .., X_1$. The state space S is the collection of all possible values of $X_1, X_2, X_3, ...$ Let $|S|$ be the total number of elements in the set S. The transition matrix T is a $|S| \times |S|$ matrix such that $[T]_{i,j} = P\{X_{n+1} = j|X_n = i\}$, where $[T]_{i,j}$ is the element in the intersection of the i-th row and the j-th column of the matrix T.

Suppose that

$$T = \begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) (5 points) What is the rank of the matrix T?

(b) (5 points) Let x be a $1 \times |S|$ matrix. How many solutions does the matrix equation $x = x \times T$ have?

(c) (10 points) Show that for each pair $(i,j) \in S \times S$, there exists an integer $n(i,j)$ such that $[T^{n(i,j)}]_{i,j} > 0$. (The value of $n(i,j)$ may depend on i and j.) You are required to find at least one $n(i,j)$ for each pair (i,j).

(d) (15 points) Does the limit $\lim_{m\to\infty} T^m$ exist? Mathematically justify your answer. (If your answer is yes, you have to derive the limit. If your answer is no, you have to prove that the limit does not exist.)

Problem 3. (Totally, 45 points) Suppose that we are interested in the price of a security as it evolves over time. Let the time starts from 0 and $S(y)$ denote the price of the security at time y, where $y \geq 0$. In general, $S(y)$ is a stochastic process. Let $\Delta > 0$ denote a small increment of time and suppose that, every Δ time units, the price of the security either goes up by a factor $u > 1$ with probability p or goes down by a factor $d < 1$ with probability $1 - p$, where

$$\begin{aligned} u &= e^{\sigma\sqrt{\Delta}} \\ d &= e^{-\sigma\sqrt{\Delta}} \\ p &= \frac{1}{2}(1 + (\frac{\mu}{\sigma}) \cdot \sqrt{\Delta}), \end{aligned}$$

where $\mu > 0$ and $\sigma > 0$ are predetermined constants.

In this problem, you are required to derive $S(y)$ step by step. For every natural number i, let Y_i be a random variable with state space $\{0, 1\}$. If the price of the security goes up at time $i\Delta$, then $Y_i = 1$. On the other hand, if the price of the security goes down at time $i\Delta$, then $Y_i = 0$. It is assumed that $Y_1, Y_2, Y_3, \ldots$ are i.i.d. random variables. Then, the number of times the security goes up in the first n time increments is $\sum_{i=1}^{n} Y_i$.

(a) (5 points) Express $S(n\Delta)$ in terms of $S(0), u, d$, and Y_i's.

(b) (5 points) Let $n = \frac{t}{\Delta}$. Express $\ln(\frac{S(t)}{S(0)})$ in terms of t, Δ, μ, σ, and Y_i's. (Recall that $\ln(\cdot)$ is the natural logarithm function.)

(c) (10 points) Derive the expectation value of $\ln(\frac{S(t)}{S(0)})$.

(d) (10 points) Derive the variance of $\ln(\frac{S(t)}{S(0)})$.

(e) (5 points) Show that as Δ approaches zero, the variance of $\ln(\frac{S(t)}{S(0)})$ approach $\sigma^2 t$.

(f) (10 points) Derive the probability density function of $\ln(\frac{S(t)}{S(0)})$ for the limit case in which Δ approaches 0.

1. (20%) Sampling Theorem
   A. Please show that $\sum_{n=-\infty}^{\infty}\delta(t-nT_0)=\frac{1}{T_0}\sum_{n=-\infty}^{\infty}e^{jn\omega_0 t}$ where $\omega_0=\frac{2\pi}{T_0}$. (Hint: Find the exponential Fourier series of a rectangular pulse train of amplitude $1/\tau$, width τ, and period T_0 for the case $\tau\rightarrow 0$.)
   B. State the sampling theorem and prove.
2. (20%) Rayleigh Distribution
   Consider a carrier signal s at a frequency ω_0 and with an amplitude a:
   $$s=a\cdot\exp(j\omega_0 t)$$
   The received signal s_r is the sum of n waves:
   $$s_r=\sum_{i=1}^{n}a_i\exp\left[j(\omega_0 t+\theta_i)\right]\equiv r\exp\left[j(\omega_0 t+\theta)\right]$$
   $$\text{where}\quad r\exp(j\theta)=\sum_{i=1}^{n}a_i\exp(j\theta_i)$$
   Define: $r\exp(j\theta)=\sum_{i=1}^{n}a_i\cos\theta_i+j\sum_{i=1}^{n}a_i\sin\theta_i\equiv x+jy$
   We have: $x\equiv\sum_{i=1}^{n}a_i\cos\theta_i$ and $y\equiv\sum_{i=1}^{n}a_i\sin\theta_i$
   where: $r^2=x^2+y^2\quad x=r\cos\theta\quad y=r\sin\theta$
   Assume (1) n is very large, (2) the individual amplitude a_i are random, and (3) the phase θ_i have a uniform distribution, it can be assumed that (from the central limit theorem) x and y are both Gaussian variables with means equal to zero and variance:
   $$\sigma_x^2=\sigma_y^2\equiv\sigma^2$$
   A. Find the PDF of the Rayleigh distribution.
   B. Find the mean value $E[R]$.
3. (20%) Matched Filter
   Prove that if a signal $s(t)$ is corrupted by AWGN, the filter with an impulse response matched to $s(t)$ maximizes the output signal-to-noise ratio. The maximum SNR obtained with the matched filter is:
   $$\text{SNR}_0=\frac{2}{N_0}\int_0^T s^2(t)\,dt=\frac{2\varepsilon}{N_0}$$
4. (15%) In a digital communications system, transmitter generates 1's and 0's according to the following rules:
   i. Coin A is a regular coin with one head and one tail.
   ii. Coin B has two heads.
   iii. The transmitter randomly chooses between coin A and coin B and

randomly put it on the table.

iv. If the transmitter sees a tail, the transmitter generates nothing and goes back to step iii.

v. If the transmitter sees a head, the transmitter turns the coin over. If the transmitter sees a head, the transmitter generates a "1". If the transmitter sees a tail, the transmitter generates a "0". Then, the transmitter goes back to step iii.

According these rules, the transmitter generates a sequence of 1's and 0's. Determine the probability of 1's.

5. (10%) Phase-Locked Loop
   A. Draw the functional blocks of a phase-locked loop.
   B. Explain the operation of the phase-lock loop.
6. (15%) Explanation
   A. Describe the Nyquist's criterion for distortionless baseband Binary Transmission.
   B. Describe the Gram-Schmidt procedure.

國立中山大學九十二學年度碩士班招生考試試題

科目：電子學　　（通訊工程研究所碩士班甲.乙組）　　

1. For the circuit shown in Fig. 1, evaluate the transfer function $T_i(s)=V_i(s)/V_s(s)$, $T_o(s)=V_o(s)/V_i(s)$ and the corresponding cutoff frequencies. (20%)

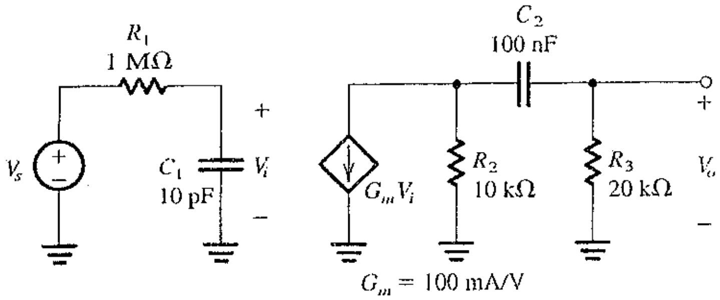

Figure 1

2. Analyze the circuit in Fig. 2 and determine the currents I_{B2}, I_E, I_{E2} and I_{C2}. Assume β=100. (20%)

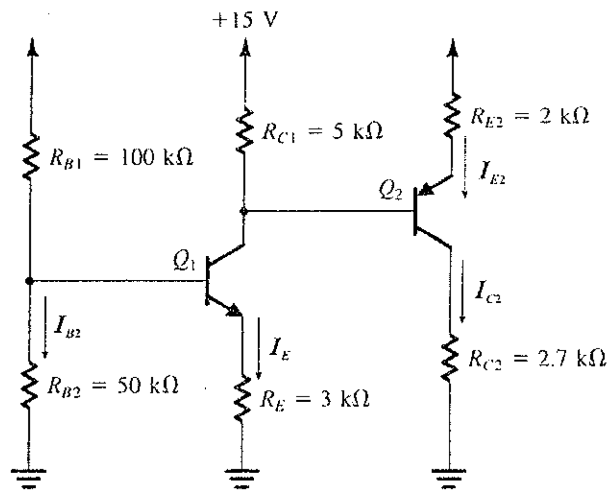

Figure 2

3. Analyze the RLC network of Fig. 3 to determine its transfer function, poles and zeros. (20%)

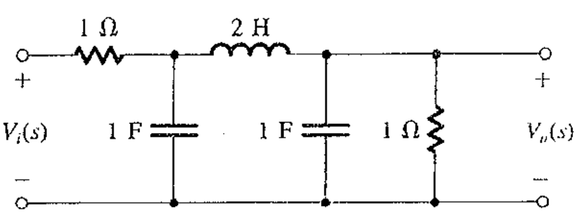

Figure 3

科目：電子學（通訊所甲.乙組）　　　　　　　　

4. Plot the transfer characteristic of the circuit in Fig. 4. (10%)

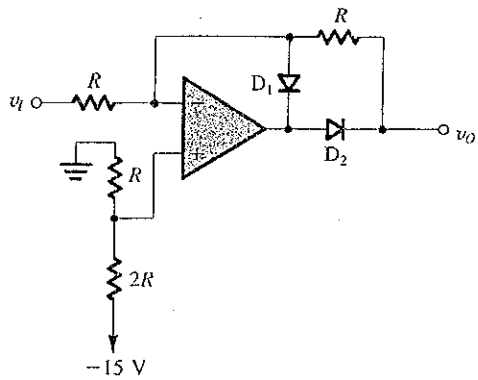

Figure 4

5. The BJTs in the circuit of Fig. 5 have β_P=10, β_N=100, $|V_{BE}|$=0.7V, $|V_A|$=100V. Find the values of the dc collector current of each transistor, V_C, v_o/v_i and R_{in}. (30%)

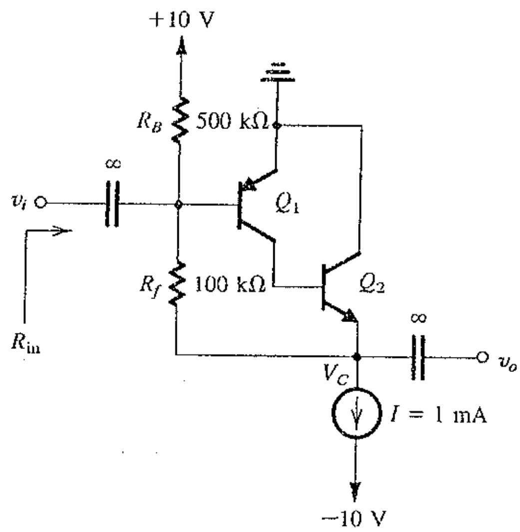

Figure 5

國立中山大學九十二學年度碩士班招生考試試題

科目：微分方程及向量分析（通訊工程研究所碩士班 乙 組）

1. (10pts) Solve the initial value problem $yy' = e^{y^2-2t}$ with $y(t=\ln 2)=0$.
2. (20pts) We would like to solve the differential equation $y'-6y=-3y^2$.
   (a) Let $y=z^k$. Rewrite the equation as $z'+pz=q$, where p is independent of z. What are p and q? (10pts)
   (b) Choose an appropriate k (5pts) and solve for $z(t)$ (5pts).
3. (20pts) We wish to solve the Laplace's equation using separation of variables
   $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ within $0\le x\le \pi$ and $0\le y\le 1$ with boundary values given by
   $u(0,y)=\frac{\partial u}{\partial x}(\pi,y)=u(x,0)=0$ and $u(x,1)=\sin(3x/2)$.
   Let $u(x,y)=X(x)Y(y)$.
   (a) Show that $X''/X=-\lambda$, where λ is a constant. (5pts)
   (b) Derive the boundary conditions for $X(\cdot)$ at $x=0$ (2pts) and $x=\pi$ (3pts).
   (c) Determine the possible λ's from the boundary conditions. (5pts)
   (d) Solve $u(x,y)$. (5 pts).
4. (30pts) The spherical coordinates are related to the rectangular coordinates through $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, $z=r\cos\theta$. Let the position vector be
   $\vec{r}=\vec{a}_x x+\vec{a}_y y+\vec{a}_z z\equiv \vec{r}(r,\theta,\phi)$.
   (a) To determine the unit vector along θ direction, $\vec{a}_\theta$, we fix r, ϕ and calculate $d\vec{r}=\vec{r}(r,\theta+d\theta,\phi)-\vec{r}(r,\theta,\phi)=\vec{a}_\theta h_\theta d\theta$. Let $\vec{a}_\theta=\vec{a}_x x_\theta+\vec{a}_y y_\theta+\vec{a}_z z_\theta$. Find $(x_\theta,y_\theta,z_\theta)$ (5pts) and h_θ (5pts) in terms of the spherical coordinates variables.
   (b) Find $\vec{a}_\phi=\vec{a}_x x_\phi+\vec{a}_y y_\phi+\vec{a}_z z_\phi$, i.e., (x_ϕ,y_ϕ,z_ϕ), in the similar manner. (5pts)
   (c) Express r in terms of rectangular coordinate variables. (5pts)
   (d) Compute $\partial^2(1/r)/\partial x^2$ (5pts) and determine if $1/r$ satisfies the Laplace's equation. Prove your assertion (5pts).
5. (10pts) Show that any solution of the equation $\nabla\times(\nabla\times\vec{A})-k^2\vec{A}=0$ automatically satisfies (a) the solenoidal condition $\nabla\cdot A=0$, (b) the vector Helmholtz equation and.
6. (10pts) Compute the integral $\oint_S \vec{r}\cdot d\vec{s}$, where $\vec{r}$ is the position vector and S encloses a cube of side length 2 (邊長為 2 之正方體) centered at the origin.

1. Consider the air-filled region enclosed on three sides by grounded conducting planes shown in Fig. 1. The conducting plane on the left is insulated from the right grounded sides and has a constant potential V_0. The gaps between the left conducting plane and the right grounded sides are assumed to be very small. In addition, all planes are assumed to be infinite in extent in the z-direction.

   (10%) (a) Determine the potential distribution within this region.

   (10%) (b) Determine the electromagnetic fields of the dominant mode in this waveguide.

   (10%) (c) Determine the characteristic impedance of the dominant mode.

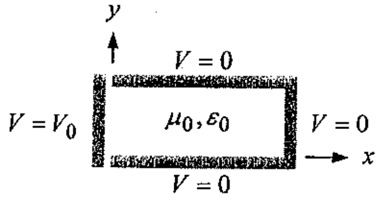

Fig. 1 Cross-sectional view of an air-filled waveguide.

2. (20%) For a time-average complex power density defined as $\vec{p}^{av} = \frac{1}{2}(\vec{E} \times \vec{H}^*)$ (W/m^2), derive that the time-average complex power flowing into a closed surface (S) can be written as $-\oint_S \vec{p}^{av} \cdot d\vec{s} = 2j\omega \int_V (w_m^{av} - w_e^{av})dv + \int_V p_\sigma^{av} dv$ where w_m^{av}, w_e^{av} and p_σ^{av} represent the time-average magnetic and electric energy densities stored and the time-average ohmic power density dissipated within the enclosed volume (V), respectively.

3. Consider a section of a uniform transmission line of length l, characteristic impedance Z_0, and propagation constant γ between terminal pairs 1-1' and 2-2' shown in Fig. 2(a). Let (V_1, I_1) and (V_2, I_2) be the phasor voltages and phasor currents at terminals 1-1' and 2-2', respectively.

   (10%) (a) For an equivalent two-port symmetrical T-network shown in Fig. 2(b), derive the expressions of the circuit elements Z_1 and Y_2 in terms of the transmission-line parameters.

   (10%) (b) For an equivalent two-port symmetrical PI-network shown in Fig. 2(c), derive the expressions of the circuit elements Y_1 and Z_2 in terms of the transmission-line parameters.

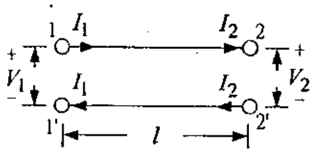

Fig. 2(a) A transmission line of length l.

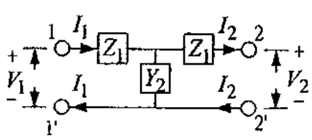

Fig. 2(b) An equivalent two-port symmetrical T-network.

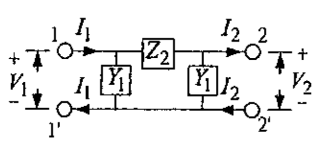

Fig. 2(c) An equivalent two-port symmetrical PI-network.

4. For a parallel-plate waveguide,

   (10%) (a) find the frequency (in terms of the cutoff frequency f_c) at which the attenuation constant due to conductor losses for the TM$_n$ mode is a minimum,

   (10%) (b) obtain the formula for this minimum attenuation constant,

   (10%) (c) calculate this minimum α_c for the TM$_1$ mode if the parallel plates are made of copper (σ=5.8$\times$10^7 S/m) and spaced 5 cm apart in air.