國立中山大學100學年度碩士班招生考試試題

科目：基礎數學【應數系碩士班甲組】

共七題。答題時，每題都必須寫下題號與詳細步驟。
請依題號順序作答，不會作答題目請寫下題號並留空白。

1. (10%) Evaluate $\lim_{n\to\infty} n(\sqrt[n]{2}-1)$.

2. (10%) Evaluate $\int_1^{64} \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$.

3. (15%) Show that $2\sin x + \cos x = x$ has three solutions in $\mathbb{R}$.

4. (15%) Calculate the area of the region Ω enclosed by the curves

$$4x^2+4xy+y^2-3x+3y=18 \quad \text{and} \quad 4x^2+4xy+y^2+3x-3y=18.$$

5. (15%) Let $(\mathbb{R}^3, <,>)$ be an inner product space with $<u,v>= u_1v_1+u_2v_2+u_3v_3$, $u=(u_1,u_2,u_3)$ and $v=(v_1,v_2,v_3)$. Let $\mathbf{V}$ be the subspace of $\mathbb{R}^3$ spanned by $(2,1,1)$ and $(1,-1,2)$. Find an orthonormal basis for $\mathbf{V}$ that contains the vector $(2/\sqrt{6}, 1/\sqrt{6}, 1/\sqrt{6})$.

6. (15%) Let A, B and $A+B$ be invertible $m\times m$ matrices. Denote the inverse matrices of A, B and $A+B$ by A^{-1}, B^{-1} and $(A+B)^{-1}$, respectively. Show that the inverse matrix of $A^{-1}+B^{-1}$ is $A(A+B)^{-1}B$.

7. (20%) Let the sequence $\{a_k\}_{k\geq 0}$ be given by $a_0=1$ and $a_{k+1}=a_k+\frac{1}{a_k}$ for $k\geq 0$. Show that the sequence $\{a_k-\sqrt{2k}\}_{k\geq 0}$ converges.

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國立中山大學100學年度碩士班招生考試試題

科目：數理統計【應數系碩士班甲組】

共八題，答題時，每題都必須寫下題號與詳細步驟。
請依題號順序作答，不會作答題目請寫下題號並留空白。

1. Suppose that the height, in inches, of a 25-year-old man is random variable with parameters $\mu = 71$ and $\sigma^2 = 6.25$. What percentage of the 25-year-old men are over 6 feet 2 inches tall? (10 分)

2. Let X and Y have a bivariate normal distribution with $\mu_X = 70$, $\mu_Y = 80$, $\sigma_X^2 = 100$, $\sigma_Y^2 = 169$, and $\rho = 5/13$. Find
   (a) $E[Y|X = 72]$. (5 分)
   (b) $\mathrm{Var}(Y|X = 72)$. (5 分)

3. If a random variable X has mean μ and standard deviation σ, show that (10 分)
$$P(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}.$$

4. If Y is $b(n, 0.25)$, give a lower bound for $P(|Y/n - 0.25| < 0.05)$ when $n = 100$. (10 分)

5. Find the Rao-Cramer lower bound if the random sample $X_1, X_2, \ldots, X_n$ is taken from the distributions having the following p.d.f.'s: $f(x;\theta) = (1/\theta^2)xe^{-x/\theta}$, $0 < x < \infty$, $0 < \theta < \infty$. (10 分)

6. Assume that the weight X in ounces of a "10-ounce" box of cornflakes is $N(\mu, 0.03)$. Let $X_1, X_2, \ldots, X_n$ be a random sample from this distribution. To test the hypothesis H_0: $\mu \geq 10.35$ against the alternative hypothesis H_1: $\mu < 10.35$, what is the critical region of size $\alpha = 0.05$ specified by the likelihood ratio test criterion? (10 分)

7. Let $X_1, X_2, \ldots, X_n$ be a random sample from $N(0, \sigma^2)$.
   (a) Find the maximum likelihood estimator for σ^2. (10 分)
   (b) Is the maximum likelihood estimator for σ^2 unbiased? (10 分)

8. Let $X_1, X_2, \ldots, X_n$ denote a random sample from Bernoulli(p).
   (a) Find the Rao-Cramer lower bound for the variance of every unbiased estimator of p. (10 分)
   (b) What is the efficiency of $\overline{X}$ as an estimator of p? (10 分)

國立中山大學100學年度碩士班招生考試試題

科目：數理統計【應數系碩士班甲組】

Table 1: 標準常態累積分佈函數：$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,dt$

$\Phi(x)$	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767

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國立中山大學100學年度碩士班招生考試試題

科目：機率論【應數系碩士班甲組】

Master Entrance Exam- Probability, 2011

(1) Assume the pair of random variables (X, Y) has a constant joint probability density function (p.d.f.), $f(x, y) = c$, on the triangle with three vertices $(0,0), (2,0)$ and $(2,1)$.

(a) Find the constant c and derive the marginal distribution of Y. (10pts)

(b) Find the conditional expectation of Y given X, that is $E(Y|X)$. (10pts)

(2) Assume (X, Y) has the multinomial density function,

$$f(x, y) = \frac{n!}{x!y!} p_1^x p_2^y$$

where x and y are nonnegative integers with $x+y = n$ and $p_1+p_2 = 1$, $p_1, p_2 > 0$.

(a) Find the marginal distribution of Y and the conditional distribution of Y given $X = x$. (10 pts)

(b) Use the result of (a) and conditional expectation to find the covariance of X and Y. (10pts)

(3) Let $Y = e^X$.

(a) Find the variance of Y, when X has the normal distribution with mean μ and variance σ^2, that is $X \sim N(\mu, \sigma^2)$. (10pts)

(b) Find the pdf of Y, when X has the following Pareto p.d.f

$$f(x) = (1 + x)^{-2}, x > 0.$$

And show that the p.d.f. of Y is symmetric. (10pts)

(4) At a party n men take off their hats. The hats are then mixed up, and each man randomly selects one. We say that a match occurs if a man selects his own hat. Let E_i denote the event that the ith man select his own hat.

(a) Find $P(E_{i1}E_{i2}\cdots E_{ik})$, where $\{i_1, i_2, \cdots, i_k\}$ is a subset of $\{1, 2, \cdots, n\}$. (10pts)

(b) Find the probability of no matches. (10pts)

(5) Assume the random variable X has the following density function:

$$f(x) = \begin{cases} (a - |x|)/a^2, & |x| \le a, \\ 0, & |x| > a. \end{cases}$$

(a) Find the moment generating function of X. (10 pts)

(b) Use the moment generating function to find the fourth moment of X, that is $E(X^4)$. (10 pts)

國立中山大學100學年度碩士班招生考試試題

科目：線性代數【應數系碩士班乙組、丙組】

1. (60 points) Let $A = \begin{bmatrix} 7 & -6 \\ 6 & -5 \end{bmatrix}$.

   (a) (10 points) Is A diagonalizable?(Give your reasons).

   (b) (10 points) Find the characteristic polynomial of A^{33}.(Give your reasons)

   (c) (10 points) Find the eigenvalues of A^{66}.(Give your reasons)

   (d) (10 points) Find the minimal polynomial of A.(Give your reasons)

   (e) (10 points) Find the minimal polynomial of A^{99}.(Give your reasons)

   (f) (10 points) Find the Jordan form of A.(Give your reasons)

2. (10 points) Prove or disprove that $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is similar.

3. (30 points) Let U, V , W be vector spaces, and let $S : U \to V$, $T : V \to W$ be linear maps. Let $C = \mathrm{Im}S \cap \mathrm{Ker}T$ (i.e. C is the intersection of Image of S and Kernel of T). Prove or disprove the followings:

   (a) (10 points) C is a vector subspace of V.

   (b) (10 points) $\dim C = \dim \mathrm{Ker}TS - \dim \mathrm{Ker}S$.

   (c) (10 points) $\dim \mathrm{Im}TS \leq \dim \mathrm{Im}S$.

國立中山大學100學年度碩士班招生考試試題

科目：微積分【應數系碩士班乙組】

計算題：共7題，子題分數平均分配。答題時，每題都必須寫下題號與詳細步驟。

[1]. (16%) Evaluate the following limits.

(a) $\lim\limits_{x\to\infty}\left(\dfrac{x}{x+3}\right)^{x}$

(b) $\lim\limits_{(x,y)\to(3,6)}\dfrac{x+y-9}{\sqrt{x+y}-3}$

[2]. (12%) Find the first derivative $F'(x)$ and the second derivative $F''(x)$ of

$$F(x)=\int_{x}^{x^2}\frac{\tan^{-1}\theta}{\theta}\,d\theta,\ \text{ where } \tan^{-1}\theta \text{ denotes the inverse function of } \tan\theta.$$

[3]. (16%) Consider the finite region in the first quadrant bounded by the curves $y=x^2$ and $y=4x$. Formulate the following quantities by integral. (**Do not need to evaluate.**)

(a) the area of the region.
(b) the volume of the solid obtained by rotating the region around the x-axis.
(c) the perimeter of the region.
(d) the volume of the solid obtained by rotating the region around the line $x=4$.

[4]. (15%) Evaluate the integral.

$$\int_0^{\infty}\frac{1}{(x+1)(x^2+1)}dx$$

[5]. (13%) Determine the interval of convergence of the power series.

$$\sum_{n=0}^{\infty}\frac{(-3)^n(x+1)^n}{\sqrt{n+1}}$$

[6]. (12%) Find the local maximal value and saddle point of $f(x,y)=6x^2-2x^3+3y^2+6xy+11$.

[7]. (16%) Consider the Cartesian integral

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\frac{4}{1+x^2+y^2}\,dxdy.$$

(a) Change the Cartesian integral into an equivalent polar integral.
(b) Evaluate the polar integral.

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國立中山大學100學年度碩士班招生考試試題

科目：數值分析【應數系碩士班乙組】

Twenty points for each problem. Please write down all the detail of your computation and answers.

1. (1)Why a subtraction of two close floating point numbers is unstable? Please give an example to illustrate it.

   (2)For very small number $\varepsilon \approx 0$, how to compute $\sqrt[3]{x+\varepsilon} - \sqrt[3]{x}$ and $\sin(x+\varepsilon) - \sin x$ to avoid unstable substraction.

2. Write a program to evaluate the polynomial

$$f(x) = \sum_{i=0}^{n} a_i x^i$$

   and its derivative at $x = c$ using least arithmetic operation. Compute the number of arithmetic operation needed in your program.

3. (1)Under what conditions will a fixed point iteration

$$x_n = g(x_{n-1}), \ n = 1, 2, \cdots$$

   converge?

   (2)Let $a > 0$. Show that the sequence

$$x_n = \frac{x_{n-1}}{2} + \frac{a}{2x_{n-1}}, \ n = 1, 2, \cdots$$

   converges to $\sqrt{a}$ for all $x_0 > 0$. What happens if $x_0 < 0$?

4. Use polynomial interpolation to prove the three-point midpoint formula for second derivative

$$f"(c) \approx \frac{1}{h^2}[f(c+h) - 2f(c) + f(c-h)],$$

   and use Taylor's Theorem to compute its error formula. Is this a stable method?

5. Write the following matrix A as the PLU decomposition $A = PLU$, where P is a permutation matrix, L lower triangular matrix, and U upper triangular matrix.

$$\begin{pmatrix} 1 & -2 & 3 & 0 \\ 1 & -2 & 3 & 1 \\ 1 & 3 & 2 & -2 \\ 2 & 1 & 3 & -1 \end{pmatrix}$$

國立中山大學100學年度碩士班招生考試試題

科目：高等微積分【應數系碩士班丙組】

Do all the problems with details. Each problem carries 20 points.

1. (a) Prove or disprove: Every bounded real sequence is convergent. [10%]

   (b) Prove or disprove: Every convergent real sequence is bounded. [10%]

2. (a) How many real solutions are there for $x^3 - 12x + 10 = 0$? [10%]

   (b) How many real solutions are there for $x = \frac{1}{2}\sin x + 3$? [10%]

3. (a) Find the extreme values of $f(x, y, z) = xy + yz$ subject to the constraints $xy = 1$ and $y^2 + z^2 = 2$. [10%]

   (b) Let $f(x, y, z) = x\sin(yz) + e^{xz} + y^2 - xy$ for $(x, y, z) \in \mathbf{R}^3$. Find the Hessian matrix of f at the point $(1, 0, 2)$. [10%]

4. Let f be a continuous function on a metric space $(\mathbf{X}, d)$, and $A \subseteq \mathbf{X}$.
   (a) Prove or disprove: If A is open, then $f(A)$ is also open. [10%]

   (b) Prove or disprove If A is compact, then $f(A)$ is also compact. [10%]

5. Let $f(x) = x\sin(\frac{1}{x})$ for $x \in (0, 1]$ and $f(0) = 0$.
   (a) Prove or disprove: f is uniformly continuous on $[0, 1]$. [10%]

   (b) Prove or disprove: f is Riemann integrable over $[0, 1]$. [10%]

End of Paper