國立中山大學103學年度碩士暨碩士專班招生考試試題

科目名稱：基礎數學【應數系碩士班甲組】 題號：424001
※本科目依簡章規定「不可以」使用計算機 共1頁第1頁

答題時，每題都必須寫下題號與步驟。

1. (10%) Evaluate $\lim_{x\to 0}\int_0^x \frac{\arctan t}{x^2}\,dt$.

2. (10%) Evaluate $\int_1^\infty e^{-x}\sqrt{x-1}\,dx$.

3. (10%) Evaluate $\int_0^\pi e^x\cos^2 x\,dx$.

4. (15%) Suppose that $\lambda \in \mathbb{R}$ and $\sum_{j=1}^4 a_{ij} = \lambda$ for $i = 1, 2, 3, 4$. Let $A = (a_{ij})$ be an 4×4 matrix. Show that A has an eigenvalue λ.

5. (15%) Calculate the area of the region Ω enclosed by the curves

$$y = \frac{1}{2+\cos x-\sin x},\quad y = \frac{\sin x}{2+\cos x-\sin x}\quad \text{and}\quad 0\le x\le \pi/2.$$

6. (20%) Find the interval of convergence of the function

$$f(x) = \left(\frac{1}{2}\right)^2 x + \left(\frac{2}{3}\right)^4 x^2 + \left(\frac{3}{4}\right)^6 x^3 + \cdots.$$

7. (20%) Find the critical points of the function $f(x,y,z) = x^2+y^2+2z^2-xy+2xz-2z$ and determine whether a relative maximum or a relative minimum occurs at each critical point.

~全卷完~

國立中山大學 103 學年度碩士暨碩士專班招生考試試題

科目名稱：線性代數【應數系碩士班乙、丙組】
※本科目依簡章規定「不可以」使用計算機

題號：424002
共 2 頁 第 1 頁

共十題，每題 10 分。答題時，每題都必須寫下題號與詳細步驟。
請依題號順序作答，不會作答題目請寫下題號並留空白。

1. Determine if the system $\begin{array}{rcrcrcrcl} x_1 & - & 6x_2 & & & & & = & 5 \\ & & x_2 & - & 4x_3 & + & x_4 & = & 0 \\ -x_1 & + & 6x_2 & + & x_3 & + & 5x_4 & = & 3 \\ & - & x_2 & + & 5x_3 & + & 4x_4 & = & 0 \end{array}$ is consistent.

2. Find the inverse of

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 2 & 2 & 0 & & 0 \\ 3 & 3 & 3 & & 0 \\ \vdots & & & \ddots & \vdots \\ n & n & n & \cdots & n \end{bmatrix}$$

3. Let $A = \begin{bmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ b & b & a & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \cdots & a \end{bmatrix}$. Find $\det A$.

4. Assume that $A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}$ is row equivalent to $B = \begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Find bases for null space and column space of A.

5. Diagonalize the matrix $\begin{bmatrix} 2 & -2 & -2 \\ 3 & -3 & -2 \\ 2 & -2 & -2 \end{bmatrix}$.

6. Assume the mapping

$$T(a_0 + a_1 t + a_2 t^2) = 3a_0 + (5a_0 - 2a_1)t + (4a_1 + a_2)t^2$$

is linear. Find the matrix representation of T relative to basis $B = \{1, t, t^2\}$.

7. Show that $I - A$ is invertible when all the eigenvalues of A are less than 1 in magnitude.

8. Suppose the eigenvalue of a 3×3 matrix A are 3, 4/5, and 3/5, with corresponding eigenvectors $\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}$, and $\begin{bmatrix} -3 \\ -3 \\ 7 \end{bmatrix}$. Let $\mathrm{x}_0 = \begin{bmatrix} -2 \\ -5 \\ 3 \end{bmatrix}$. Find the solution of the equation $\mathrm{x}_{k+1} = A\mathrm{x}_k$ for the specified x_0, and describe what happens as $k \to \infty$.

9. If B is invertible prove that AB has the same eigenvalues as BA.

背面有題

10. For what numbers c and d are A and B positive definite?

$$A = \begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & d & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

國立中山大學103學年度碩士暨碩士專班招生考試試題

科目名稱：高等微積分【應數系碩士班丙組】　　　　　　　　　　　　題號：424003

※本科目依簡章規定「不可以」使用計算機　　　　　　　　　　　　　共1頁第1頁

Do all the problems with details. Each problem carries 20 points.

1.

(a) Find $\liminf \left[(-1)^n + \frac{1}{n}\right]$ and $\limsup \left[(-1)^n + \frac{1}{n}\right]$. [10%]

(b) Prove or disprove: If every subsequence of $\{a_n\}$ converges to the same limit, then $\{a_n\}$ converges. [10%]

2.

(a) Find the interval of convergence of $\sum_{n=2}^{\infty} n(n-1)x^n$. [10%]

(b) Find the sum in (a) for x in the interval of convergence. [10%]

3.

(a) Prove or disprove: Every subset of a compact set in the Euclidean space $\mathbf{R}^n$ is also compact. [10%]

(b) Let f be a continuous function on $\mathbf{R}$. Is the set $\{x \in \mathbf{R} | f(x) > 0\}$ open? closed? or neither? [10%]

4. Let f_n be a sequence of continuous functions on $[0,1]$. Suppose that $f_n(x) \to f(x)$ for each $x \in [0,1]$.

(a) Prove or disprove: f is continuous on $[0,1]$. [10%]

(b) Prove or disprove: $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 f(x)\,dx$. [10%]

5.

(a) Evaluate the line integral $\int_C y\,dx + x\,dy$, where C is the curve $x = 2t$, $y = e^t$, $0 \le t \le 1$. [10%]

(b) Evaluate the surface integral $\int\int_\Omega y d\sigma$ where Ω is the surface $z = x + y^2$, $0 \le x \le 3$, $0 \le y \le 1$. [10%]

End of Paper

國立中山大學103學年度碩士暨碩士專班招生考試試題

科目名稱：微積分【應數系碩士班乙組】　　　　　　　　題號：424004

※本科目依簡章規定「不可以」使用計算機　　　　　　　　共1頁第1頁

計算題：共8題，子題分數平均分配。答題時，每題都必須寫下題號與詳細步驟。

[1]. (12%) Give the definitions of the limit of functions, the continuity of functions, the derivative of functions, and the limit of sequences respectively.

[2]. (14%) Evaluate the following limits.

(a) $\lim_{x\to\infty}\frac{x+3\sin x}{2x+5-7\sin 2x}$　　　(b) $\lim_{x\to 0^+}\frac{1}{x}\int_0^{2x}\frac{\sin 3t}{t}dt$

[3]. (20%) Evaluate the following integrals.

(a) $\int_{-\infty}^{0} xe^{-|x|}dx$　　　(b) $\int\frac{12}{\sqrt{9-8x-x^2}}dx$

[4]. (12%) Find the maximal and minimal distances between the origin and the ellipse $x^2+xy+y^2=12$.

[5]. (12%) Find positive p values for which the series $\sum_{n=2}^{\infty}\frac{\ln n}{n^p}$ converges.

[6]. (12%) Evaluate the following iterated integral.

$$\int_0^1\int_y^1\frac{1}{1+x^4}dxdy$$

[7]. (10%) Solve the differential equation $y'=-y+x$ with $y(2)=2$.

[8]. (8%) Air is being pumped into a spherical balloon at a rate of 2.5 cubic centimeters per second. Find the rate of change of the radius when the radius is 3 centimeters.

========= 全卷完 =========

國立中山大學103學年度碩士暨碩士專班招生考試試題

科目名稱：數理統計【應數系碩士班甲組】
※本科目依簡章規定「不可以」使用計算機

題號：424005
共1頁 第1頁

注意事項：

1. 本試卷共五大題，每大題 20 分。答題時，每題都必須寫下題號與詳細步驟。
請依題號順序作答，不會作答題目請寫下題號並留空白。

Below are some well-known facts that you might need to use to answer the following questions.
i.i.d.: identically independently distributed. CDF: cumulative distribution function. Denote $\Phi(t)$ as the CDF for the standard normal random variable. $\Phi(0.9) = 1.28$, $\Phi(0.95) = 1.65$, $\Phi(0.975) = 1.96$.

1. Tom spins a coin three times and observes no heads. Then he gives the coin to Marry. Marry spins it until the first tail occurs, and ends up spinning it four times total. Let p denote the probability the coin comes up heads.
   (a) (10%) What is the likelihood of p?
   (b) (10%) What is the MLE of p?

2. Let $X_1, \ldots, X_n$ be i.i.d. random variables with CDF F (i.e. $F(t) = P(X_1 \le t)$). Define the empirical CDF $F_n(t) = \frac{\sum_{i=1}^n 1_{\{X_i \le t\}}}{n}$, where 1 is the indicator function. Please answer the following questions.
   (a) (10%)Prove $F_n(t)$ is an unbiased estimator for $F(t)$.
   (b) (10%)If n is large, construct an asymptotic 95% confidence interval for $F(t)$ using $F_n(t)$. (You have to give your reasoning)

3. Answer the following questions:
   (a) (10%)Let X be a random variable with distribution exp(1). Find a constant c so that it minimizes $f(c) = E(X^4 - c)^2$.
   (b) (10%)You throw a fair dice (6 sides) once. Let the number you observe be X. Therefore, X can be $1, 2, 3, 4, 5$, or 6 with probability $1/6$ for each possible value. Define Y as 1 if X is a multiple of 3 and as 0 otherwise. Find $g(Y)$ that minimizes $E\left[(X - g(Y))^2 | Y\right]$

4. Under H_0, a random variable has the CDF $F_0(x) = x^2, 0 \le x \le 1$; and under H_1, it has the CDF $F_1(x) = x^3, 0 \le x \le 1$.
   (a) (10%)What is the form of the likelihood ratio test of H_0 versus H_1?
   (b) (10%)What is the rejection region of a level α test?

5. (20%) $X_1, \ldots, X_n$ are i.i.d. random variables (r.v.) with Cauchy(θ) distribution (i.e. probability density function $f(x)$ of X_1 is $\frac{c}{1+(x-\theta)^2}$, where c is a known constant). Please find the minimal sufficient statistic for θ. (You have to prove the statistic you give is minimal sufficient)

國立中山大學103學年度碩士暨碩士專班招生考試試題

科目名稱：數值分析【應數系碩士班乙組】　　題號：424006

※本科目依簡章規定「不可以」使用計算機　　共1頁第1頁

Please write down all the detail of your computation and answers.

1. (20%) (1) Give the definition that a numerical method is stable.

   (2) Give an example to show the subtraction of two close floating point numbers is unstable.

2. (20%) Choose any two different methods to compute the cubic polynomial $p(x)$ interpolating the following data

x	-1	0	1	2
y	-5	-1	3	13

3. (20%) Write down the algorithms of secant method, method of false position (regula falsi) and the fixed point method to solve a root γ of the nonlinear equation $f(x) = 0$.

4. (20%) Use polynomial interpolation to prove the midpoint formula for first derivative

$$f'(c) \approx \frac{1}{2h}[f(c+h) - f(c-h)]$$

   and to obtain its error formula. Is this a stable method?

5. (20%) Write a Gaussian elimination program to solve $n \times n$ linear system $A\mathbf{x} = \mathbf{b}$ that needs no row exchange. Compute its operation counts for additions/subtractions and multiplications/divisions.

國立中山大學 103 學年度碩士暨碩士專班招生考試試題

科目名稱：機率論【應數系碩士班甲組】 題號：424007
※本科目依簡章規定「不可以」使用計算機 共 1 頁 第 1頁

共十題，每題 10 分。答題時，每題都必須寫下題號與詳細步驟。
請依題號順序作答，不會作答題目請寫下題號並留空白。

1. In how many ways can n identical balls be distributed into r urns so that the ith urn contains at least m_i balls, for each $i = 1, \ldots, r$? Assume that $n \geq \sum_{i=1}^{r} m_i$.

2. Balls are randomly removed from an urn initially containing 20 red and 10 blue balls. What is the probability that all of the red balls are removed before all of the blue ones have been removed?

3. Let $S = \{1, 2, \ldots, n\}$ and suppose that A and B are, independently, equally likely to be any of the 2^n subsets (including the null set and S itself) of S. Find $P(A \subseteq B)$.

4. There are k types of coupons. Independently of the types of previously collected coupons, each new coupon collected is of type i with probability p_i, $\sum_{i=1}^{k} p_i = 1$. If n coupons are collected, find the expected number of distinct types that appear in this set.

5. If X is uniformly distributed over $(0, 1)$, find the density function of $Y = e^X$.

6. If X has hazard rate function $\lambda_X(t)$, compute the hazard rate function of aX where a is a positive constant.

7. The joint density of X and Y is

$$f(x, y) = c(x^2 - y^2)e^{-x} \quad 0 \leq x < \infty,\ -x \leq y \leq x$$

Find the conditional distribution of Y, given $X = x$.

8. If X and Y are independent random variables both uniformly distributed over $(0, 1)$, find the joint density function of $R = \sqrt{X^2 + Y^2}$, $\Theta = \tan^{-1} Y/X$.

9. The joint density of X and Y is given by

$$f(x, y) = \frac{e^{-y}}{y} \qquad 0 < x < y,\ 0 < y < \infty$$

Compute $E[X^3 | Y = y]$.

10. Let $X_1, \ldots, X_n$ be independently and identically distributed with pdf $f_X(x)$. What is

$$\lim_{n \to \infty} \left(\prod_{i=1}^{n} X_i \right)^{1/n}$$