國立中山大學 106 學年度碩士暨碩士專班招生考試試題

科目名稱：基礎數學【應數系碩士班甲組】　　　　題號：424001

※本科目依簡章規定「不可以」使用計算機(問答申論題)　　　　共1頁第1頁

共十題，每題 10 分。答題時，每題都必須寫下題號與詳細步驟。
請依題號順序作答，不會作答題目請寫下題號並留空白。

1. 設 $a_n = \sqrt{1\times 2} + \sqrt{2\times 3} + \cdots + \sqrt{n(n+1)}$，求 $\lim\limits_{n\to\infty} \frac{a_n}{n^2}$ 之值爲?

2. 已知座標平面上點 $P_n(x_n, y_n)$ 滿足 $(x_{n+1}, y_{n+1}) = (7x_n + 3y_n, 3x_n + 7y_n)$，$n$ 爲非負的整數，其中 $(x_0, y_0) = (1+a, 1-a)$, $a \in \mathbb{R}$，且 $a \neq 0$。若 O 表座標平面的點，試求 $\lim_{n\to\infty} \frac{2}{n} \log \overline{OP_n}$ 之值。

3. 計算 $\sum_{k=1}^{20} k^4$ 之值。

4. 設 $f(x) = \dfrac{\prod\limits_{k=0}^{50}(x-2k)}{\prod\limits_{k=1}^{50}(x+k)}$，求 $\log_2 f'(0)$ 值。

5. 若直線 $y = 3x + a$ 與曲線 $y = x^3 + 2$ 有三相異交點，求 a 的範圍。

6. 若曲線 $y = 2x - x^2$ 與 x 軸所圍部分面積被直線 $y = mx$ 二等分，則實數 m 之值爲何？

7. 試證：半徑爲 r 的球體的體積爲 $\frac{4}{3}\pi r^3$。

8. 矩陣 $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \\ -1 & 1 & 2 \end{bmatrix}$，若 $A^3 + aA^2 + bA + cI = O$，求 $a + b + c$。

9. 令 Ω 爲直線 $y - x = 1,\ y - x = 2,\ 2x + y = 0,\ 2x + y = 2$ 所圍成的區域。計算 $\iint_\Omega (y-x)(2x+y)\,\mathrm{d}A$。

10. 求函數 $f(x, y) = xy$ 在曲線 $x^2 + xy + y^2 = 1$ 上之最大值，最小值及其所在之點。

科目名稱：機率與統計【應數系碩士班甲組】　　題號：424006

※本科目依簡章規定「不可以」使用計算機(問答申論題)　　共1頁第1頁

答題時，每題須寫下題號與詳細步驟。請依題號順序作答，不會作答題目請寫下題號並留空白。

Notation:
i.i.d.: identically independently distributed; pdf: probability density function; MLE: maximum likelihood estimator; $Exp(\theta)$ random variable means a random variable with exponential distribution with a parameter θ and its pdf is $f(x|\theta) = \frac{1}{\theta}e^{-\frac{x}{\theta}}, x > 0$. $\bar{X}$ is the sample mean.

1. (15%) $X_1, \ldots, X_n$ are i.i.d. random variables with mean μ and variance σ^2. Assume $\mu \neq 0$. What is the asymptotic distribution of
$$\sqrt{n}(\frac{1}{\bar{X}} - \frac{1}{\mu}) + \frac{\sum_{i=1}^n X_i^2}{n}?$$
Express your answer using μ and σ.

2. (15%) A stick of unit length is randomly cut into two pieces. Please compute the expected ratio of the length of the shorter piece to that of the longer piece.

3. (15%) Let $X_1, \ldots, X_n$ be i.i.d. $Exp(1)$ random variables. Define the random variables $Y_1, \ldots, Y_n$ by $Y_i = 1$ if $X_i \geq 3$ and $Y_i = 0$ if $X_i < 3$. Find the distribution of $\sum_{i=1}^n Y_i$.

4. (20%) Let $X_1, \ldots, X_n$ be $Exp(\theta_1)$ random sample and $Y_1, \ldots, Y_n$ $Exp(\theta_2)$ random sample. $X's$ and $Y's$ are independent.
Find the likelihood ratio test of $H_0 : \theta_1 = \theta_2$ against $H_a : \theta_1 \neq \theta_2$ with significance level α (You have to specify the test statistic and the rejection region).

5. (20%) $X_1, \ldots, X_n$ are i.i.d. Bernoulli(p) random variables. Let $q = P(X_1 = 1, X_2 = 1)$. Find the uniformly minimal-variance unbiased estimator (UMVUE) of q.

6. (15%) Let $X_1, \ldots, X_n$ be a random sample from a continuous distribution F (no ties). The distribution of F has mean μ and variance σ^2. Let $X_1^*, \ldots, X_n^*$ be the sample with replacement randomly chosen from $X_1, \ldots, X_n$. Let $\bar{X}^* = \frac{\sum_{i=1}^n X_i^*}{n}$. Please Calculate $\mathrm{Var}(\bar{X}^*)$ (Express your answer using σ and n).

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科目名稱：微積分【應數系碩士班乙組】 題號：424002

※本科目依簡章規定「不可以」使用計算機(問答申論題) 共1頁第1頁

計算題：共7題，子題分數平均分配。答題時，每題都必須寫下題號與詳細步驟。

[1]. (14%) Let

$$f(x) = \begin{cases} \dfrac{\sin(x^2)}{x^2}, & \text{for } x \neq 0, \\ 1, & \text{for } x = 0. \end{cases}$$

(a) Is $f(x)$ continuous at $x = 0$? Why?

(b) Is $f(x)$ differentiable at $x = 0$? Why?

[2]. (14%)

(a) Let $f(x) = 3^x \sin\left(\frac{3\pi}{4}x\right)$. Find the equation of the tangent line at $x = 2$.

(b) Let $f(x, y) = 2e^{-(x+1)^2-(y-2)^2}$. Find the equation of the tangent plane at $(-1, 1)$.

[3]. (16%) Let $a, b, c > 0$. Use any method in calculus to show that

(a) the area of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab.

(b) the volume of an ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is $\frac{4}{3}\pi abc$.

[4]. (12%) Determine the interval of convergence for the power series $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n3^{n+1}}$.

[5]. (20%) Evaluate the following integrals.

(a) $\int_2^{\infty} (x-2)^2 e^{-x} dx$ (b) $\int_{-1}^{0} \frac{x^3 - 1}{x^2 - 2x + 1} dx$

[6]. (12%) Solve the differential equation $y' + 2y = e^{3x}$ with $y(0) = 1$.

[7]. (12%) Show that the Laplace equation $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ in the coordinates (r, θ), where $x = r\cos\theta$ and $y = r\sin\theta$, is

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = 0.$$

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國立中山大學 106 學年度碩士暨碩士專班招生考試試題

科目名稱：線性代數【應數系碩士班乙組】　　　　題號：424005
※本科目依簡章規定「不可以」使用計算機(問答申論題)　　　　共1頁第1頁

Please write down all the detail of your computation and solution.

1. (15%) Find a, b, c, d such that the linear system

$$\begin{cases} x+y+2z=b \\ 2x+ay-2z=c \\ 3x+6y+3z=d \end{cases}$$

has (1) a unique solution, (2) infinitely many solutions, (3) no solution.

2. (15%) Consider the planar transformation F reflects a vector about the line $x+y=0$, and then rotates it 30° about the origin. Find the inverse transformation of F.

3. (15%) Find the projection matrix onto the plane $x-y+z=0$ in $\mathbf{R}^3$.

4. (15%) Let A be an $m\times n$ real matrix and $\mathbf{b}$ be an n dimensional real column vector. Show that exactly one of the following statements holds: (1) $A\mathbf{x}=\mathbf{b}$ has a solution $\mathbf{x}$, (2) $A^t\mathbf{y}=\mathbf{0}$ has a solution $\mathbf{y}$ such that $\mathbf{y}^t\mathbf{b}\neq 0$.

5. (20%) Let A be an $n\times n$ matrix. Prove A is diagonalizable if and only if A has n linearly independent eigenvectors.

6. (20%) Let A be an $n\times n$ nonsingular matrix and B be an $n\times m$ matrix. State the fastest numerical method to compute (1) determinant of A, (2) $A^{-1}B$. How many arithmetic operations are needed in (1), and what is the minimal memory needed in (2)?

國立中山大學106學年度碩士暨碩士專班招生考試試題

科目名稱：線性代數【應數系碩士班丙組】　　　　　　　　　　　　　　題號：424003

※本科目依簡章規定「不可以」使用計算機(問答申論題)　　　　　　　　　共1頁第1頁

共六道題。答題時，每題須寫下題號與詳細步驟。 請依題號順序作答，不會作答題目請寫下題號並留空白。

1. [10%] Let $\{v_1, v_2, v_3\}$ be a linearly independent set in some vector space. Show that the set $\{v_1 + v_2, v_2 + v_3, v_3 + v_1\}$ is also linearly independent.

2. [30%] For any $n \in \mathbb{N}$, denote by $P_n(\mathbb{R})$ the collection of polynomials over $\mathbb{R}$ with degrees less or equal to n. Define a map $T : P_3(\mathbb{R}) \to \mathbb{R}^3$ as

$$T(p(x)) = (p(-1), p(0), p(1))$$

for $p \in P_3(\mathbb{R})$.

(1) [10%] Show that $P_3(\mathbb{R})$ is a vector space.

(2) [10%] Show that T is linear.

(3) [10%] Find bases for the kernel $N(T)$ and range $R(T)$ of T, respectively.

3. [15%] Let T be the linear transform on the set $M_n(\mathbb{R})$ of $n \times n$ matrices over $\mathbb{R}$ defined as $T(A) = A^t$, the transpose of $A \in M_n(\mathbb{R})$. Find all eigenvalues of T.

4. [15%] Find the Jordan form of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

5. [15%] Let A be an $m \times n$ matrix with rank m. Prove that there exists an $n \times m$ matrix B so that $AB = I_m$, the $m \times m$ identity matrix.

6. [15%] Let V be a vector space with dimension n. Suppose that $T : V \to V$ is a linear transform and that there exists some vector $v \in V$ satisfying $T^{n-1}v \neq 0$ and $T^n v = 0$. Show that T admits the matrix representation

$$\begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$$

with respect to some basis of V.

End of Paper

國立中山大學106學年度碩士暨碩士專班招生考試試題

科目名稱：高等微積分【應數系碩士班丙組】　　題號：424004
※本科目依簡章規定「不可以」使用計算機(問答申論題)　　共1頁第1頁

答題時請標示題號並詳列計算及推導過程. 每小題10分.

1. Let $a_1 = 2$ and $a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n}$ for $n = 1, 2, 3, \ldots$.

   (a) Show that $\{a_n\}$ is a decreasing sequence.

   (b) Does $\{a_n\}$ converge? If yes, find the limit.

2. Let $f(x) = x^2 \sin(\frac{1}{x})$ for $x \neq 0$ and $f(0) = 0$.

   (a) Is f uniformly continuous on $(-1, 1)$?

   (b) Is f differentiable on $(-1, 1)$?

3. Let $f(x) = e^{-\frac{1}{x^2}}$ for $x \neq 0$ and $f(0) = 0$.

   (a) Is f differentiable at $x = 0$?

   (b) Is f analytic at $x = 0$?

4. Let A and B be compact subsets of a metric space $(\mathbf{X}, d)$.

   (a) Is $A \cap B$ compact? Why?

   (b) Is $A \cup B$ compact? Why?

5. (a) Find the absolute maximum and minimum values of $f(x, y) = x^2 - 2x + y^2 - 2y + 8$ on the disk $x^2 + y^2 \leq 9$.

   (b) Evaluate the line integral $\int_C (x^3\,dx + y\,dy)$ where C is the curve $y = x^2 + 1$ from $(0, 1)$ to $(2, 5)$.

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