（橫書式）

國立中山大學八十七學年度碩博士班招生考試試題

科 目：(應數所)基礎數學 A 甲組

Do *all* the problems in detail.

(1) (i) Evaluate $\int_0^1 \frac{x^2+1}{x^4+1}\,dx$. [10%]

(ii) Determine whether the series $\sum_{n=1}^{\infty} \frac{n^2 n!}{n^n}$ converges or diverges. [10%]

The Gamma function is defined for $x > 0$ as

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}\,dt.$$

(iii) Show that $\Gamma(x+1) = x\Gamma(x)$. [10%]

(iv) Find $\Gamma(2.5)$. [10%]

(2) (i) Find a real matrix B such that $A = B^3$ where

$$A = \begin{pmatrix} 9 & -7 \\ -7 & 9 \end{pmatrix}.$$

[20%]

(ii) Let $\mathbf{V}$ be the vector space of 2×2 matrices over $\mathbf{R}$, and let $M = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$. Let T be the linear operator on $\mathbf{V}$ defined by $T(A) = MA$. Find the trace of T. [10%]

(iii) Let $P = (p_{ij})$ be an $n \times n$ real matrix with $p_{ij} \geq 0$ for $i, j = 1, 2, \ldots, n$ and $\sum_{j=1}^{n} p_{ij} = 1$ for each $i = 1, 2, \ldots, n$. Show that P has an eigenvalue 1. [10%]

(3) (i) Let

$$F(x) = \begin{cases} 0, & x < 0 \\ 2, & 0 \leq x < 2 \\ 5, & 2 \leq x < 3 \\ 6, & x \geq 3 \end{cases}.$$

Evaluate the Riemann-Stieltjes integral

$$\int_{-10}^{10} e^{-sx}\,dF(x).$$

[10%]

(ii) Let f be a continuous function from a metric space $\mathbf{X}$ into a metric space $\mathbf{Y}$, and let A be a compact subset of $\mathbf{X}$. Is $f(A)$ compact in $\mathbf{Y}$? Why? [10%]

End of Paper

1. If the joint probability density of X_1 and X_2 is given by

$$f(x_1, x_2) = \begin{cases} e^{-(x_1+x_2)} & \text{for } x_1 > 0, x_2 > 0 \\ 0 & \text{otherwise.} \end{cases}$$

find the probability density of $Y = X_1/(X_1 + X_2)$ and the variance of Y. (15%)

2. Let X_n and $Y_{m,n}$ be t distribution with degree of freedom n and F distribution with degree of freedom m and n.
   (a) Write down the definitions of X_n and $Y_{m,n}$ in terms of normal and Chi-Square random variables. (10%)
   (b) Explain why $X_n \to N(0,1)$ in distribution as $n \to \infty$ and $Y_{1,n} = X^2$. (10%)

3. Let X_i be independent normal distributed with mean $\beta_0 + \beta_1 x_i$ and variance σ^2 where x_i's are fixed constants and $\beta_0, \beta_1, \sigma^2$ are unknown parameters.
   (a) Find the maximum likelihood estimators for β_0, β_1 and σ^2. (10%)
   (b) Find the Fisher information matrix for β_0, β_1 and σ^2. (10%)
   (c) Are the maximum likelihood estimators for β_0, β_1 and σ^2 uniformly minimum variance unbiased estimators? (10%)

4. In a random sample, 136 of 400 persons given a flu vaccine experienced some discomfort. Construct a 95% confidence interval for the true proportion of persons who will experience some discomfort from the vaccine. (10%)

5. Given a random sample of size n from a normal population with unknown mean and variance, find an expression for the likelihood ratio statistic for testing the null hypothesis $\sigma = \sigma_0$ against the alternative hypothesis $\sigma \neq \sigma_0$. (15%)

6. A single observation is to be used to test the null hypothesis that the mean waiting time between tremors recorded at a seismological station (the mean of an exponential population) is $\theta = 10$ hours against the alternative that $\theta \neq 10$ hours. If the null hypothesis is to be rejected if and only if the observed value is less than 8 or greater than 12, find the probabilities of type I error and the type II error when $\theta = 2$. (10%)

So faith comes out of hearing, and hearing through the word of Christ.
– Roman 10:17

1. 設機車騎士有高（佔人數 20%），中（佔 30%）及低（佔 50%）三個危險群。某保險公司估計在某一年度內三個危險群每人至少有一件以上意外事故的機率分別為 0.25, 0.16, 0.10. 問

(a) 任選一騎士，求其在當年內至少有一件以上意外事故的機率. 6%

(b) 在某一意外事故現場發現一騎士. 問此騎士來自高危險群的機率有多少？ 6%

2. 設 C 表一圓，$\triangle XYZ$ 為其上之內接正三角形. 在此圓上任取一弦 $\overline{UV}$，求 $\overline{UV}$ 長度大於正三角形邊長的機率.
（註：此題有許多不同的解答方式，答案也不一樣. 只要分析合理即可）. 10%

3. 設 $X>0$ 為一隨機變數，平均值為 μ.

(a) 証明 $\mu=\int_0^\infty p\{X\geq x\}dx$ 8%

8% (b) 証明 $p\{X\geq c\mu\}\leq\frac{1}{c}$ $\forall c>0$.（稱為 Markov 不等式）

4% (c) 舉一例說明 Markov 不等式與 Chebyshev 不等式 $p\{|X-\mu|\geq k\sigma\}\leq\frac{1}{k^2}$ 的優劣（不限定優或劣，只要舉一例可分出好壞即可）.

4. 設符号 $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ 表來自於 X 的 order statistics（由小至大）. 請回答以下問題：

(a) 若 X 的機率密度函數為 $f(x,\lambda)=\frac{1}{\lambda}\exp(-\frac{x}{\lambda})$, $\lambda>0$, $x\geq 0$.
令 $W_i=(n-i+1)(X_{(i)}-X_{(i-1)})$, $i=1,2,\cdots,n$, $X_{(0)}=0$.
証明 $W_1, W_2, \cdots, W_n$ 為 i.i.d 且求其分佈. 10%

(b) 若 X 為 uniform $U(0,1)$, 機率密度函數為

$$f(x)=\begin{cases}1 & x\in(0,1).\\ 0 & \text{其它}\end{cases}$$

令 $Y_1=\frac{X_{(1)}}{X_{(2)}}$, $Y_2=\frac{X_{(2)}}{X_{(3)}}$, $\cdots$, $Y_{n-1}=\frac{X_{(n-1)}}{X_{(n)}}$, $Y_n=X_{(n)}$. 証明 $Y_1, Y_2, \cdots, Y_n$ 為独立, 且求各個 Y_i 的分佈. 10%

5. 設 $X_1, X_2, \cdots$ 為 i.i.d $P\{X_n=\pm 1\}=\frac{1}{2}$, $n=1,2,\cdots$ 令 $Z_n=\sum_{k=1}^{n}\frac{X_k}{2^k}$.
要証明 $Z_n \xrightarrow[n\to\infty]{\mathcal{D}} U(-1,1)$. 亦即 Z_n 的分佈在 $n\to\infty$ 時會接近 uniform $U(-1,1)$. 請依以下二階段証明此結論：

(a) 証明 Z_n 的 moment generating function $M_n(t)=E(e^{tZ_n})$ 為

$$M_n(t)=\frac{\sinh t}{2^n\sinh\left(\frac{t}{2^n}\right)} \quad \left(\text{其中 } \sinh t=\frac{e^t-e^{-t}}{2}\right)$$

（提示：重複使用 $2\sinh t\cosh t=\sinh(2t)$） 10%

(b) 証明 $\lim_{n\to\infty} M_n(t)=\frac{e^t-e^{-t}}{2t}$, 此即為 $U(-1,1)$ 的 m.g.f. 10%

6. 敘述強大數法則 (Strong Law of Large Numbers) 及中央極限定理 (Central Limit Theorem). 8%

7. 請由標準常態分佈的簡單知識不必查表，請大約估計若投擲一枚均勻硬幣 10000 次則出現正面的次數低於 4850 次的機率。 10%

（横書式）

國立中山大學八十七學年度碩博士班招生考試試題

科 目：基礎數學B 乙組（應用數學系） 共 1 頁 第 1 頁

1 (16 points). Let W be the subspace of R^4 spanned by $[1,0,1,0]$, $[1,1,1,0]$ and $[1,-1,0,1]$.

(a): Find an orthonormal basis for W;

(b): Find the projection of $[1,1,1,1]$ on W.

2 (14 points). Find the eigenvalues λ_i and the corresponding eigenvectors v_i of the linear transformation T on R^3 defined by

$$T([x_1,x_2,x_3]) = [x_1, 4x_2+7x_3, 2x_2-x_3].$$

3 (12 points). Prove that if A is a square matrix, then the nullity of A is the same as the nullity of A^T.

4 (10 points). Let G be a graph. Prove that $\chi(G) \leq \Delta(G)+1$. ($\Delta(G)$ denotes the maximum degree of G.)

5 (12 points). Suppose the minimum degree of graph G is $\delta(G)$. Prove that G contains a cycle of length at least $\delta(G)+1$.

6 (12 points). How many integral solutions are there for the following inequalities:

$$x_1+x_2+x_3+x_4 < 100, \quad x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 > 0, \quad x_4 > 0.$$

7 (12 points). Solve the recurrence relation:

$$x_n = x_{n-1} + x_{n-2}, \quad x_0 = x_1 = 1.$$

8 (12 points) Prove any graph G contains a bipartite subgraph H such that $|E(H)| \geq |E(G)|/2$.

1. If a computer will be connected to the Internet, some network settings have to be done, such as IP address, host name, domain name, gateway, subnet mask and etc.

   (a) What is a gateway? What is its purpose? (3%)

   (b) What is the purpose of the subnet mask? What is the relationship between the subnet mask and the IP address? Please explain with an example. (4%)

   (c) What is a subnetwork of Class A in the Internet? What is a subnetwork of Class B in the Internet? What is the relationship between the IP address and Class A or B? It would be better if you can give some examples to explain. (4%)

   (d) The data transmission protocol used in an Ethernet LAN is CSMA/CD (more precisely, IEEE 802.3 CSMA/CD). How does an Ethernet LAN resolve transmission collision? (4%)

2. Please explain each of the following terms. (18%)

   (a) indexed sequential file.

   (b) NP-complete.

   (c) B-tree.

   (d) FreeBSD.

   (e) Postscript.

   (f) 3-to-8-line decoder.

3. What is the concept of memory interleaving? And what is the advantage of memory interleaving? (6%)

4. (a) What is addressing mode? (3%)

   (b) Give three addressing modes and give examples to explain them. (6%)

5. Simplify the Boolean expression

$$(A \bigoplus B) \bigoplus (A \bigoplus D) \bigoplus (C \bigoplus E) \bigoplus (D \bigoplus E) \bigoplus (B \bigoplus D)$$

   where $\oplus$ denotes the Exclusive OR (XOR) operation. (5%)

6. In a computer system with multiple CPU's, an interconnection network may be used to interconnect those CPU's. And an interconnection network can be modeled as a graph $G = (V, E)$, in which V is the set of nodes, each representing a CPU, and E is the set of edges, each representing a link connecting two CPU's. The hypercube is one of the famous interconnection networks. The n-dimensional hypercube, denoted as Q_n, consists of 2^n nodes in which each node is identified by a bit string of length n. Suppose the identifier of a node is $b_{n-1} \cdots b_1 b_0$, then it will be connected to each node with identifier $b_{n-1} \cdots \overline{b_i} \cdots b_1 b_0$, $0 \le i \le n-1$. In other words, there is an edge between node $b_{n-1} \cdots b_1 b_0$ and node $b_{n-1} \cdots \overline{b_i} \cdots b_1 b_0$. For instance, in Q_5, node 11010 is connected to node 01010, node 10010, node 11110, and etc. If there is an edge between node x and node y, it is said that one *routing step* is needed to send data from node x to node y, or from node y to node x. Note that some *intermediate nodes* may be used to send data from some node to another node if they are not connected by an edge directly. The *distance* between two nodes is the number of routing steps between them.

(a) Draw the hypercube Q_3. Each node has to be labeled with its identifier.(3%)

(b) Does there exist a 3-level complete binary tree in Q_3? Explain your reason. Note that in a complete binary tree, the root is on level 1, the two sons of the root are on level 2 and the four grandsons of the root are on level 3. Thus, a 3-level complete binary tree contains 7 nodes. (5%)

(c) In Q_5, how do you send data from node 11010 to node 00001? That is, what are the intermediate nodes from between node 11010 and node 00001? Note that there may be more than one method to send data from node 11010 to node 00001. You need give only one of the methods. (3%)

(d) In Q_n, what is the largest distance between all pairs of nodes? Why? (4%)

7. Let $S_n = \{1, 2, \cdots, n\}$. An m-permutation of S_n is obtained by selecting m distinct integers out of the n and arranging them in some order. Now let $x = (x_1\ x_2\ \cdots\ x_m)$ and $y = (y_1\ y_2\ \cdots\ y_m)$ be two m-permutations of S_n. We say that x precedes y in *lexicographic order* if there exists an i, $1 \leq i \leq m$, such that $x_j = y_j$ for all $j < i$ and $x_i < y_i$. And the *rank* of an m-permutation p, denoted as $r(p)$, is its position in the lexicographical order of all m-permutations. For example, all 3-permutations of S_4 in lexicographical order are (1 2 3), (1 2 4), (1 3 2), (1 3 4), (1 4 2), (1 4 3), (2 1 3), $\cdots$, (4 3 2). And $r(1\ 2\ 3) = 1$, $r(1\ 4\ 2) = 5$, $r(4\ 3\ 2) = 24$ and etc. Answer the following questions for the 4-permutations of S_7.

(a) What is the next one of (5 3 4 7)? (3%)

(b) Which permutation has the rank 828? Explain how do you calculate? (5%)

(c) What is $r(5\ 3\ 4\ 7)$? Explain how do you calculate? (6%)

(d) Which permutation has the rank 530? Explain how do you calculate? (6%)

8. Write a C or Pascal program to find the maximum number stored in the nodes of a binary tree. Note that in the binary tree, each node stores one integer. You can use the following declaration in your C program. (12%)

```
sturct nodetype {
    int info;
    struct nodetype *left;
    struct nodetype *right;
}
```

Or, you can use the following declaration in your Pascal program.

```
type nodeptr= ↑ nodetype;
     nodetype=record;
        info: integer;
        left: nodeptr;
        right: nodeptr;
     end;
```

You have to state explicitly which language you are using to write the program.

1.(10%)
Compare the differences between macro expansion and subroutine call in terms of (a) memory space, (b) CPU time, (c) parameter and argument, (d) binding time, (e) register.

2.(8%)
Define and expain 4 tables needed for a two-pass assembler, including when they are created, when they are used and the contents.

3.(6%)
Compare the differences among the (a) linking loader, (b) linking editor, (c) dynamic linking.

4. (8%)
List four essential components of a grammar and describe their limitation.

5. (8%)
Write a BNF (or EBNF) grammar which can accept the expression A + B * C **D.

6. (12%)
Assume you have the following jobs to execute with one processor:

Job	Burst-Time	Priority
1	6	5
2	2	1 (highest)
3	3	3
4	1	4
5	5	2

The jobs are assumed to have arrived in the order 1,2,3,4,5.
(1) Give a Grantt chart illustrating the execution of these jobs using First-come-first-served, round-robin (quantum=1), shortest-job-first and a non-preemptive priority scheduling algorithms.
(2) What is the turnaround time of each job for each of the above scheduling algorithms ?
(3) What is the waiting time of each job for each of the above scheduling algorithms ?

7. (10%)
Consider the following page reference string:
1,2,3,4,2,1,5,6,2,1,2,3,7,6,3,2,1,2,3,6.
How many page faults would occur for the following replacement algorithms, assuming 3,4,5,6 or 7 frames ?
(a) LRU.
(b) FIFO.

8. (8 %)
A microcomputer uses the buddy system for memory management. Initially, it has one block of 256K at address 0. After successive requests for 5K, 25K, 35K, and 20 K come in, how many blocks are left and what are their sizes and addresses ?

9. (10 %) How to use monitors to "simulate" P and V operations of a semaphore ?

10. (10%)
Assume you have a page reference string for a process with M frames (initially all empty). The page reference string has length P with N distinct page numbers occuring in it. For any page replacement algorithms,
(a) What is the lower bound on the number of page faults ?
(b) What is an upper bound on the number of page faults ?

11. (10%)
How long does it take to load a 64K program from a disk whose average seek time is 30 msec, whose rotation time is 20 msec, and whose tracks hold 32K,
(a) for a 2Kpage size ?
(b) for a 4K page size ?
The pages are spread randomly around the disk.

國立中山大學八十七學年度碩~~博~~士班招生考試試題 丙組

科 目:(應用數學系碩士班)高等微積分 共 1 頁 第 1 頁

ANSWER *all* OF THE FOLLOWING QUESTIONS, EACH OF WHICH CARRIES 20 OUT OF 100 POINTS.

1. (a) Evaluate the double integral
$$\iint_D xy,$$
where D is the closed triangular area in the xy-plane with vertices $(0,0)$, $(0,1)$ and $(1,0)$.

   (b) Compute the line integral
$$\int_\Gamma xy\,dx + (x+y)\,dy,$$
where Γ is the boundary of the triangle with vertices $(0,0)$, $(0,1)$, and $(2,0)$.

2. Find all possible local maximum, local minimum and saddle points of the function
$$f(x,y) = x^2 - xy + y^2 + 2x + 2y - 4,$$
and evaluate the values of f at these points.

3. (a) Use the method of *Lagrange's multipliers* to find the least distance of the point $(1,2,3)$ to the plane
$$4x + 5y + 6z = 7.$$

   (b) Find and classify the extreme values (if any) of the function
$$f(x,y) = y^2 + x^3.$$

4. (a) Show that the set $\mathbb{R}^{n-1} \times \{0\}$ has measure zero in $\mathbb{R}^n$.

   (b) Let $[0,1]^2 = [0,1]\times[0,1]$. Let $f : [0,1]^2 \longrightarrow \mathbb{R}$ be defined by setting $f(x,y) = 0$ if $y \neq x$ and $f(x,y) = 1$ if $y = x$. Show that f is integrable over $[0,1]^2$.

5. Show that Simpon's Rule gives the exact result on any interval $[\alpha - h, \alpha + h]$ when applied to a polynomial $f(x) = ax^3 + bx^2 + cx + d$ of degree 3 or less.

End of Paper

Solve all the problems in detail.

1. Evaluate the integral, $k \in \mathbf{R}$, $a > 0$,
$$\int_{-\infty}^{\infty} \frac{e^{-ikx}}{x^2 + a^2}\, dx.$$
[15%]

2. Evaluate for $t > 0$,
$$\lim_{L \to \infty} \int_{2-iL}^{2+iL} \frac{e^{zt}}{z^4}\, dz.$$
[15%]

3. Determine whether the limit $\lim_{z \to 0} (z/\bar{z})^2$ exists. [15%]

4. Let f and g be analytic at a point z_0. Suppose that $f(z_0) \neq 0$, $g(z_0) = 0$, and $g'(z_0) \neq 0$. Show that z_0 is a simple pole of the quotient $f(z)/g(z)$, also find the residue of $f(z)/g(z)$ at z_0. [15%]

5. Find the Laurent series (or Taylor series) expansions for
$$f(z) = \frac{1}{(z-1)(z-3)}$$
in the domain (i) $|z| < 1$, (ii) $1 < |z| < 3$. [20%]

6. Find a linear fractional transformation f which mapps the upper half plane $\operatorname{Im} z > 0$ onto the open disk $|z| < 1$ and the line $\operatorname{Im} z = 0$ onto the unit circle $|z| = 1$, also $f(1) = 1$. [20%]

End of Paper

註：每題十分，無證明或計算過程者，不予計分。

1. Find the Jordan form of matrix $A = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}(-1, 2, -1)$.

2. Factor matrix $\begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 2 \end{pmatrix} = QR$, where Q is an orthogonal matrix and R upper triangular matrix.

3. Find a matrix with trace 1, determinant -2, and eigenvectors $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

4. Show that square matrices A and A^t have the same eigenvalues. Assume A is diagonalizable, A and A^t have the same eigenvectors, prove that A is symmetric.

5. Let A be an $m \times n$ real matrix and $\mathbb{b}$ an column vector in $\mathbb{R}^m$. Show that exactly one of following statements holds:
① $A\mathbb{x} = \mathbb{b}$ has a solution $\mathbb{x}$, ② $\begin{cases} A^T \mathbb{y} = \mathbb{0} \\ \mathbb{y}^T b \neq 0 \end{cases}$ has a solution $\mathbb{y}$.

6. Let A be an $m \times n$ matrix with rank r. Find the relation between m, n, r if for every $\mathbb{b}$, $A\mathbb{x} = \mathbb{b}$ has ① infinitely many solutions, ② exactly one solution.

7. Find the necessary and sufficient condition on c such that matrix $\begin{pmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{pmatrix}$ is symmetric positive definite.

8. Let $F(x, y) = (x+y, x-y)$, find its matrix representation with respect to new basis which is obtained by rotating standard basis 60° counterclockwise about the origin.

9. Factor matrix $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 7 & 7 \\ 2 & 7 & 9 \end{pmatrix} = LL^T$, where L is a lower triangular matrix.

10. Given integers $m < n$ and n data $(x_1, y_1), \cdots, (x_n, y_n)$ with at least $m+1$ number of $x_1, \cdots, x_n$ are distinct. Prove that there is a unique least square polynomial of degree m fit to these data.

1. 計算 $\sin x$，其中 $x=0.1$，準確到五位有效位數（十進位）。

2. 對于非線性方程求根，推導牛頓（Newton）迭代法和割線（Secant）迭代法，並比較他們之優劣。

3. 設已知$(n+1)$個點 (x_i, y_i), $i=0,1,\cdots,n$，且 x_i 兩兩不相等。拉格朗日（Lagrange）插值公式為

$$y=\sum_{i=0}^{n} y_i L_i(x), \quad n \geq 1. \qquad (1)$$

問基函數 $L_i(x)$ 是甚麼？並証明 (1) 式嚴格通過$(n+1)$個點，且有等式成立：

$$\sum_{i=0}^{n} L_i(x)=1, \qquad \sum_{i=0}^{n} x_i L_i(x)=x.$$

4. 設函數 $f(x,y)$ 具有二階偏導數，寫出計算二維重積分

$$\int_0^1\int_0^1 f(x,y)\,dx\,dy$$

之合成中点（Composite Central）公式，並推導出它的誤差界。

5. 求解常微分方程初值問題

$$\begin{cases} y' = f(x,y), & a \leq x \leq b \\ y|_{x=a} = y_0. \end{cases} \qquad (2)$$

(接上頁)

寫出求解(2)式的梯形(Trapezodial)公式，並給出誤差分析与穩定性分析(Error analysis and stability analysis).

註：每題20分，滿分100分.

1. [**memory system**] Suppose that the elements of an array is stored in memory in *row major* order by the compiler. Write an efficient program, in a virture storage environment, to initialize an $n \times n$ array A to identity matrix, where n is a large number. Explain why your program is efficient. (10%)

2. [**numerical computation**] Define *well-conditioned* and *ill-conditioned* problems. Can the precision of the computations of an ill-conditioned problem be improved? Justify your answer. (10%)

3. [**job scheduling**] What are *preemptive* and *nonpreemptive* scheduling? Which scheduling is more suitable for a multi-user computer? Justify your answer. (10%)

4. [**parameter passing**] Explain call by value, call by reference, call by name and call by value-result (call by copy), and show what will be the results of the following PASCAL-like program for each of the four parameter passing methods. Give detail explanations for your answers. (25%)

```
program pass(input, output);
    var i: integer; A: array[1..2] of integer;
    function S(var x: integer): integer;
        begin
            x := x + 1;
            S := x;
        end;
    begin
        A[1] := 0; A[2] := 0;
        i := S(A[S(A[1])]);
        writeln(i)
    end.
```

5. Determine "true" or "false" for each of the following statements, and justify your answers.
   (a) [**number representation**] A positive integer n requires no more than $\lceil \log_2 n \rceil$ bits in its binary representation, where $\lceil \log_2 n \rceil$ is the smallest integer greater than or equal to $\log_2 n$. (5%)
   (b) [**programming language**] The statement $x = x + 1$ should never be used in any FORTRAN program, since there are no solutions to the equation. (5%)
   (c) [**algorithm**] There is no efficient way to find the best route for a traveling salesman who wants to visit all the major cities in Taiwan, even if the set of cities is given, because the traveling salesman problem is NP-complete. (5%)

6. [**algorithm design**] Let $a_1, a_2, \ldots, a_n$ be a sequence of n integers. Design an efficient algorithm to determine if there is an element $a_i = i$ under each of the following assumptions.
   (a) Assume that all integers are between 1 and n. (10%)
   (b) Assume that $a_1 < a_2 < \ldots < a_n$. (10%)
   (c) Assume that $0 < a_1 < a_2 < \ldots < a_n$. (10%)

每題 20分，滿分 100分

1. 記向量范數 $\|x\|_p = \left(\sum_{i=1}^{n} x_i^p\right)^{1/p}$，$p \geq 1$，$n \geq 1$。

証明下面兩不等式成立。

$$\frac{1}{\sqrt{n}}\|x\|_1 \leq \|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\,\|x\|_2, \qquad (1)$$

$$\frac{1}{n}\|x\|_1 \leq \|x\|_\infty \leq \|x\|_1 \leq n\,\|x\|_\infty. \qquad (2)$$

2. 設矩陣 $A \in \mathbb{R}^{n\times n}$，有代數方程組

$$Ax = b, \quad x \in \mathbb{R}^n, \quad b \in \mathbb{R}^n \qquad (3)$$

(a) 敘述方程(3)有唯一解的充分必要条件，並証明之。

(b) 敘述方程(3)有多值解的充分必要条件。

3. 設矩陣 $A \in \mathbb{R}^{n\times n}$，$B \in \mathbb{R}^{n\times n}$，$C \in \mathbb{R}^{n\times n}$，$A$，$B$ 和 C 為對稱(Symmetric)矩陣，B 為正定矩陣，C 為正半定矩陣。証明：

(a) 矩陣 $B^{-1}A$ 和 $(B+C)^{-1}A$ 的最大特徵值(λ_{max})和最小特徵值(λ_{min})均為實數。

(b) 且有不等式

$$\lambda_{min}\{(B+C)^{-1}A\} \leq \lambda_{min}\{B^{-1}A\}, \qquad (4)$$

$$\lambda_{max}\{(B+C)^{-1}A\} \le \lambda_{max}\{B^{-1}A\}. \quad (5)$$

4. 設 矩陣 $A\in\mathbb{R}^{n\times n}$，且行列式 $|A|\neq 0$.

敘述高斯消去法 (Gaussian Elimination) 求解方程組.

$$AX=b, \quad x\in\mathbb{R}^n, \quad b\in\mathbb{R}^n.$$

又設向量 b 有誤差 Δb，有

$$A(x+\Delta x)=b+\Delta b.$$

推導出相對誤差 (Relative errors)，$\frac{\|\Delta x\|}{\|x\|}$，之上界，其中 $\|x\|$ 是歐氏 (Euclieari) 範數

$$\|x\| = \|x\|_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}.$$

5. 設有矛盾方程組 (Overdetermined system)

$$Ax=b, \quad A\in\mathbb{R}^{m\times n}, \quad x\in\mathbb{R}^n, \quad b\in\mathbb{R}^m, \quad m>n \quad (6)$$

且知秩為 n (Rank(A) = n). 給出求解 (6) 的方法，及其穩定性分析 (Stability analysis).

—— End ——