Do *all* the problems in detail.

(1) (i) Find the interval of convergence of $\sum_{n=1}^{\infty} n^{-1}2^{-n}(x-1)^n$. [10%]

(ii) Find the volume in the first octant bounded by the three coordinate planes and the surface $z=(1+x+3y)^{-3}$. [10%]

(iii) Show that f is continuous at a if and only if $\forall x_n \to a$, $f(x_n) \to f(a)$. [20%]

(2) Let A and B be $n \times n$ matrices.

(i) Let $tr(A)$ denote the trace of A. Show that $tr(AB) = tr(BA)$. [10%]

(ii) Define $A \sim B$ if there is an invertible P such that $A = PBP^{-1}$. Show that if $A \sim B$ then $tr(A) = tr(B)$. [5%]

(iii) Let V be a finite dimensional vector space, and $E : V \to V$ be a projection operator, i.e., $E^2 = E$. Show that E is diagonalizable. [15%]

(3) (i) Suppose f is a real function defined on $\mathbf{R}$ which satisfies

$$\lim_{h\to 0^+}[f(x+h)-f(x-h)] = 0$$

for every $x \in \mathbf{R}$. Is f continuous on $\mathbf{R}$? Prove your answer. [15%]

(ii) Is $f(x,y) = (xy + x^2 + 2, x + y)$ a one-to-one function in some open neighborhood of $(1,1)$? If yes, find the derivative of its inverse at $(4,2)$. [15%]

End of Paper

ANSWER *all* 5 QUESTIONS, EACH OF WHICH CARRIES 20 POINTS.

1. Find the length of the polar curve
$$r = \sqrt{1+\cos 2\theta}, \quad 0 \le \theta \le \pi\sqrt{2}.$$

2. Suppose that $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = 2$. Find the radii of convergence of the following series.
   (a) $\sum_{n=0}^{\infty} a_n^2 x^n$.
   (b) $\sum_{n=0}^{\infty} a_n x^{2n}$.
   (c) $\sum_{n=0}^{\infty} a_n x^{n^2}$.

3. (a) Let f be continuous at a and $f(a) > 0$. Prove that there is a $\delta > 0$ such that $f(x) > 0$ for all x in $(x-\delta, x+\delta)$.
   (b) Let f be monotone (increasing or decreasing) on $[a,b]$. Prove that f is integrable on $[a,b]$.

4. Let A be the set in $\mathbb{R}^2$ defined by the equation
$$A = \{(x,y) \mid x > 1 \text{ and } 0 < y < 1/x\}.$$
Calculate $\int_A 1/xy^{1/2}\, dxdy$ if exists.

5. Let $I = [0,1]$ and $Q = I \times I$. Define $f : Q \longrightarrow \mathbb{R}$ by
$$f(x,y) = \begin{cases} \frac{1}{p} & \text{if } y \text{ is rational and } x = \frac{q}{p}, p, q \in \mathbb{N} \text{ such that} \\ & \text{the greatest common factor } (p,q) \text{ of } p \text{ and } q \text{ is } 1, \\ 0 & \text{otherwise.} \end{cases}$$
   (a) Is f integrable over Q? If yes, compute $\int_Q f$.
   (b) Compute $\underline{\int}_{y\in I} f(x,y)$ and $\overline{\int}_{y\in I} f(x,y)$.
   (c) Show that $\int_{y\in I} f(x,y)$ exists for x in $I - D$, where D is a set of measure zero in I.
   (d) Verify Fubini's theorem for $\int_Q f$.

End of Paper

Entrance Exam for the Master Program of Scientific Computing

Six questions with the marks indicated.

1.(15) Simply state the ideas of stability of numerical methods, address the differences between stability and convergence, and provide examples to briefly explain them.

2.(15) Suppose that there exists a root of $f(x) = 0$, and $0 \leq m \leq f'(x) \leq M$. Prove that

$$x_{n+1} = x_n - \lambda f(x_n), \quad n = 0, 1, \ldots$$

yields the convergent sequence $\{x_n\}$ to the root for arbitrary $x_0 \in (-\infty, \infty)$ and $0 < \lambda < 2/M$.

3.(15) Let

$$A \in R^{n \times n}, \ Cond.(A) = \{\frac{\lambda_{max}(A^T A)}{\lambda_{min}(A^T A)}\}^{1/2},$$

where $\lambda_{max}(A)$ and $\lambda_{min}(A)$ are the maximal and minimal eigenvalues of matrix A, respectively. Prove

(1). $Cond.(AB) \leq Cond.(A) Cond.(B)$,

(2) $Cond.(UA) = Cond(A)$, where $U \in R^{n \times n}$ is an orthogonal matrix.

4.(15) Let $Ax = b, \ A\tilde{x} = \tilde{b}$, where $det|A| \neq 0$. Prove

$$\frac{||x - \tilde{x}||}{||x||} \leq ||A|| ||A^{-1}|| \frac{||b - \tilde{b}||}{||b||}$$

where $x \neq 0$, $b \neq 0$ and $||x||$ is any vector norm.

5.(20) Let $f(x) \in C^2(a, b)$. Prove

$$\int_a^b f(x)dx = \frac{3h}{2}[f(a+h) + f(a+2h)] + \frac{3h^3}{4} f''(\xi), \quad a < \xi < b, \quad h = \frac{b-a}{3}.$$

6.(20) Write down the five point-difference scheme for solving

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial x^2} = f(x,y) \ \ in \ \ S, \qquad u = g(x,y) \ \ on \ \ \partial S,$$

with the unit square $S : \{0 \leq x \leq 1, \ 0 \leq y \leq 1\}$. Provide the relaxation iteration method and describe the optimal relaxation parameter.

==================== Good luck ===================

1. 設 $U_1, U_2, \cdots$ 為 i.i.d $U(0,1)$, uniform 分佈

(a) 證明 $X_i = (-\log U_i)/\lambda$ 為 exponential 分佈參數為 λ.

(b) 設 N 為一隨机變數 其值為 $0, 1, 2, \cdots$
定義 $N = n$ iff $\prod_{i=1}^{n} U_i \geq e^{-\lambda} > \prod_{i=1}^{n+1} U_i$ （20分）
求 N 的分佈.

(c) 若 $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ 為 $X_1, X_2, \cdots, X_n$ 的 order statistics.
定義 $W_1 = n X_{(1)}$, $W_2 = (n-1)(X_{(2)} - X_{(1)})$, $\cdots$, $W_k = (n-k+1)(X_{(k)} - X_{(k-1)})$
$k = 1, 2, \cdots, n$. 求各個 W_i 的分佈.

(d) 如何從 $X_1, X_2, \cdots, X_n$ 估計 λ. 請找一個並說明它為何是最好的。

2. 設 $X_1, X_2, \cdots \sim$ i.i.d $P(X_i = 1) = P(X_i = -1) = \frac{1}{2}$

(a) 估計 $P\{|\overline{X}_{1000}| \geq 0.04\}$ $\overline{X}_{1000} = \sum_{i=1}^{1000} X_i / 1000$.

(b) 用 moment generating function 或 characteristic function 的方法求 Y 的分佈, 其中 Y 定義為

$$Y = \sum_{i=1}^{\infty} \frac{X_i}{2^i}$$ （20分）

3. 美國有一家大銀行, 它所雇用的系統工程師在撰寫電腦軟体時動了手腳. 他將每一個客戶在每次計算利息之後只要是"分"的部份, 全部都跑進他的帳户裡面. 這個犯罪行為在五年之後才被揭發。請問若此銀行一共有500萬個帳户, 每個月計息一次, 則此人獲不法利益美金一億元以上之機率為何？（詳述你的步驟及所使用的統計理論）。（20分）

4. 設 $X_1, X_2 \sim i.i.d\ P(\lambda)$ (Poisson) 請回答以下問題

(a) 請問：何謂 Sufficient Statistics？

(b) $T(X_1,X_2)=X_1+X_2$ 是否為 sufficient Statistics？ (20分)

(c) $T^*(X_1,X_2)=X_1+2X_2$ 是否為 sufficient Statistics？

(d) Sufficient Statistics 有什麼用？

5. 觀察一個樣本，想要知道此樣本是來自

H_0: $X \sim \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ 或來自

$x \in R$

H_1: $X \sim \frac{1}{2} e^{-|x|}$

(20分)

請找一個 most powerful size α test.

詳細說明你的步驟及統計理論

國立中山大學八十八學年度碩博士班招生考試試題

科　目：應數系碩士班甲機率論　　　　共 1 頁 第 1 頁

(1) Let X and Y be independent and uniformly distributed on $[0,1]$.

(b) Find the probability density of $Z = \frac{X}{Y}$. [15%]

(b) Find the probability that the roots of the quadratic equation $t^2 + Xt + Y = 0$ are real. [10%]

(2) There are 5 cards: two are red on both sides, two are white on both sides, and one has a white side and a red side. A card is picked at random, i.e., with equal probability, and then pick one of its side at random. The side is white. What is the probability that the other side is also white? [10%]

(3) Box A contains 5 red and 10 black balls. Box B contains 10 red and 5 black balls. Toss a fair coin: If Heads, move a random ball from Box A to Box B. If Tails, move a random ball from Box B to Box A. Now toss the coin twice (and move 2 balls). What is the probability that the color distribution will be the same as it was at the start? [15%]

(4) Suppose that X and Y are i.i.d. Poisson random variables with parameter λ. Is $\frac{X+Y}{2}$ Poisson? Why or why not? [10%]

(5) Let X, Y be independent uniform random variables on $\{1, 2, \ldots, N\}$

(a) Find the probability density of $Z = |X - Y|$. [15%]

(b) Find the probability generating function of X. [10%]

(6) Let X and Y be independent exponential with parameter λ. Prove or disprove: $X + Y$ and X must be independent. [15%]

1. 令

$$A = \begin{pmatrix} 1 & 1 & 1 & -3 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 2 & -3 \\ 2 & 2 & 4 & -5 \end{pmatrix}$$

求 A^{-1}. (10分)

2. 設 A 是 n 階方陣，$n>1$。用 $\operatorname{adj}A$ 表示 A 的伴隨矩陣。

證明：

(a) $r(A) = n \iff r(\operatorname{adj}A) = n$

(b) $r(A) = n-1 \iff r(\operatorname{adj}A) = 1$

(c) $r(A) < n-1 \iff r(\operatorname{adj}A) = 0$

其中 $r(B)$ 表示矩陣 B 的秩. (30分)

3. 在空間 R^3 上，求 k 個平面 $a_i x + b_i y + c_i z + d_i = 0$, $1 \le i \le k$, 相交於一點的充要條件？相交於一線的充要條件？(10分)

4. 求下列行列式值：(15分)

$$\begin{vmatrix} (2n-1)^n & (2n-2)^n & \cdots & n^n & (2n)^n \\ (2n-1)^{n-1} & (2n-2)^{n-1} & \cdots & n^{n-1} & (2n)^{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ (2n-1) & (2n-2) & \cdots & n & 2n \\ 1 & 1 & \cdots & 1 & 1 \end{vmatrix}$$

5. 求下列方陣的 Jordan canonical form：

$$B = \begin{pmatrix} 0 & 1 & -2i & 0 \\ 0 & 0 & 0 & -2i \\ 2i & 0 & 0 & 1 \\ 0 & 2i & 0 & 0 \end{pmatrix}$$

其中 $i = \sqrt{-1}$。(15分)

6. 設 V 為一佈於實數系的向量空間且 $\dim V = n$。設 $f: V \to V$ 為一線性映射，且存在向量 $a \in V$ 以及某個正數 k，具有性質：$f^{(k-1)}(a) \neq 0$ 且 $f^{(k)}(a) = 0$。請證明：

$$a,\ f(a),\ \cdots,\ f^{(k-1)}(a)$$

線性獨立。(20分)

Solve all the following problems in detail.

In the following, we shall denote the set of all complex numbers by $\mathbf{C}$.

Problem 1. Let $f(z) = |z|^2$ for all $z \in \mathbf{C}$.

(a) Is f differentiable at the origin ? Prove your answer. (5 %)

(b) Is f analytic at the origin ? Prove your answer. (5 %)

Problem 2. The chordal distance $\sigma(z, w)$ between any two points z and w of the complex plane is defined by

$$\sigma(z, w) = \frac{2|z-w|}{\sqrt{1+|z|^2} \cdot \sqrt{1+|w|^2}}.$$

Let $U \subset \mathbf{C}$ be non-empty. If for every $z \in U$ there is a real number $r_z > 0$ such that $\mathbb{B}(z, r_z) = \{\zeta \in \mathbf{C} : \sigma(\zeta, z) < r_z\} \subset U$, prove that U is an open subset of the complex plane. (10 %)

Problem 3. Let $\gamma(\theta) = 2e^{i\theta}$ for $0 \le \theta \le 2\pi$. Evaluate the line integrals :

(a) $\displaystyle\int_\gamma \frac{\cos \pi z}{(z-1)^3}\, dz$. (10%)　　(b) $\displaystyle\int_\gamma \frac{e^z}{z(z-1)^3}\, dz$. (10 %)

Problem 4. Let $\gamma : [0,1] \longrightarrow \{z \in \mathbf{C} : |z-2| < 2\}$ be a smooth curve with $\gamma(0) = 2$ and $\gamma(1) = 1 + i$. Evaluate the line integral :

$$\int_\gamma \frac{dz}{z}. \qquad (10\%)$$

Problem 5. Let Ω be a non-empty open subset of $\mathbf{C}$, and let $\{f_n\}_{n=1}^\infty$ be a sequence of complex valued functions continuous on Ω. Assume that $\{f_n\}_{n=1}^\infty$ converges to a function f uniformly on every compact subset of Ω.

(a) Prove that f is continuous on Ω. (8 %)

(b) If $\gamma : [0,1] \longrightarrow \Omega$ is a piecewise smooth curve, prove that

$$\lim_{n\to\infty} \int_\gamma f_n(z)\, dz = \int_\gamma f(z)\, dz. \qquad (7\ \%)$$

Problem 6. Let $f : \mathbf{C} \longrightarrow \mathbf{C}$ be analytic. If

$$\lim_{z\to\infty} f(z) = \infty,$$

prove that there is a $z_0 \in \mathbf{C}$ such that $f(z_0) = 0$. (10 %)

Problem 7. Let T be a Möbius transformation (or linear fractional transformation). Assume that T maps the circle $|z-1| = 1$ onto the circle $|z+2| = 1$, and that $T(0) = -1$ and $T(\frac{1}{2}) = 0$.

(a) Find the transformation T. (8 %)

(b) If $C = \{T(z) : |z-4| = 1\}$, describe C geometrically and explicitly. (7 %)

Problem 8. Let $D = \{z \in \mathbf{C} : |z| \le 1\}$, and let f be a complex valued function analytic on an open set containing D. If $|f(z) - 1| < 1$ for all z with $|z| = 1$, prove that $f(z) \neq 0$ for all $z \in D$. (10 %)

國立中山大學八十八學年度碩~~博~~士班招生考試試題

科 目： 計算機概論 （應用數學系,碩士班,丁組） 共 1 頁 第 1 頁

1. **[number representation]** How to represent a fixed point number and a floating point number in a computer? Why do we need to distinguish a fixed point number from a floating point number? (10%)

2. **[storage]** What are *little-ending* and *big-ending*? (10%)

3. **[memory system]** What is *virtual memory*? Why do we need virtual memory? (10%)

4. **[memory system]** Suppose that the elements of an array is stored in memory in *row major* order by the compiler you are going to use. Write an efficient program, in a virture storage environment, to initialize an $n \times n$ array A to identity matrix, where n is a very large number. Explain why your program is efficient. (10%)

5. **[numerical computation]** Define *well-conditioned* and *ill-conditioned* problems. Give an example for well-conditioned problem and an example for ill-conditioned problem. Can the precision of the computation of an ill-conditioned problem be improved? (10%)

6. **[parameter passing]** Explain *call-by-reference*, *call-by-value*, *call-by-name*, and *call-by-result*. Give an example to show that call-by-reference and call-by-value may produce different results. (10%)

7. Determine "true" or "false" for each of the following statements, and justify your answers.

   (a) **[number representation]** A positive integer n requires no more than $\lceil \log_2 n \rceil$ bits in its binary representation, where $\lceil \log_2 n \rceil$ is the smallest integer greater than or equal to $\log_2 n$. (5%)

   (b) **[programming language]** The statement $x = x + 1$ should never be used in any FORTRAN program, since there are no solutions to the equation. (5%)

   (c) **[algorithm]** There is no efficient method to find the best route for a traveling salesman who wants to visit all the major cities in Taiwan, even if the set of cities is given, because the traveling salesman problem is NP-complete. (5%)

8. **[programming language]** Determine whether the following statement is true or false for the programming languages C, FORTRAN, PASCAL and LISP. (10%)

   Since all procedure calls are first-in-last-out, all memories allocated for the local variables of a procedure can be released when that procedure exits.

9. **[algorithm design]** Write a program, in BASIC, C, FORTRAN, or PASCAL, to find the greatest common divisor of two positive integers a and b by using the Euclidean algorithm. Show that your program must terminate for any positive integers a and b. (15%)

國立中山大學八十八學年度碩博士班招生考試試題

科　目：線性代數（應數系丁組）共 1 頁 第 1 頁

Answer all 5 questions. Each carries 20 points.

(1) Find the general matrix representation for the reflection of the plane in the line $y = mx$.

(2) Find the bases of the row space, column space and nullspace of the following matrix A, respectively.

$$A = \begin{pmatrix} 1 & 3 & 0 & -1 & 2 \\ 0 & -2 & 4 & -2 & 0 \\ 3 & 11 & -4 & -1 & 6 \\ 2 & 5 & 3 & -4 & 0 \end{pmatrix}.$$

(3) Let

$$A = \begin{pmatrix} -4 & 6 & -12 \\ 3 & -1 & 6 \\ 3 & -3 & 8 \end{pmatrix}.$$

(i) Find the characteristic polynomial.

(ii) Find the real eigenvalues and the corresponding eigenvectors.

(iii) Find an matrix C and a diagonal matrix D such that $D = C^{-1}AC$.

(4) Let $f : V \to W$ and $g : V \to W'$ be linear transformations such that $\ker g \subseteq \ker f$. Show that there exists a linear function $h : W' \to W$ such that $h \circ g = f$. (Hint. Consider extending a basis of $\ker g$ to a basis of V and remember that $\dim V = \dim(\ker g) + \dim(\operatorname{Im} g)$.)

(5) Let $y_0, y_1, y_2, \ldots$ be the sequence of the Fibonacci numbers where $y_0 = 0$, $y_1 = 1$ and $y_{n+1} = y_n + y_{n-1}$ for all $n \geq 2$. Let $z_n = y_{n-1}$ for $n \geq 1$. Then the Fibonacci sequence can be written as a first order recurrences system

$$y_{n+1} = y_n + z_n,$$
$$z_{n+1} = y_n$$

with initial conditions $y_1 = 1$ and $z_1 = 0$. By setting $\mathbf{y}_n = (y_n, z_n)^t$ and $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, one obtain

$$\mathbf{y}_{n+1} = A\mathbf{y}_n.$$

Now, diagonalize A and obtain a formula for the $(n+1)$-th Fibonacci number y_n.