ANSWER *all* 5 QUESTIONS, EACH OF WHICH CARRIES 20 POINTS.

1. Let

$$A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$$

   (a) Find the real eigenvalues and the corresponding eigenvectors of A.

   (b) Find an invertible matrix C and a diagonal matrix D such that $D = C^{-1}AC$.

2. Show that every 2×2 orthogonal matrix is of one of two forms: either

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \quad or \quad \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

for some angle θ. What is the geometric meaning of this fact?

3. Compute, with justification of each step, the integral

$$I = \int_{-\infty}^{+\infty} e^{-x^2}\, dx.$$

4. Compute the line integral

$$\oint_\Gamma xy\, dy - y^2\, dx,$$

where Γ is the boundary of the triangle with vertices $(0,0)$, $(0,1)$, and $(2,0)$ with counter-clockwise orientation.

5. (a) Use the method of *Lagrange's multipliers* to find the least distance of the point $P(x,y,z)$ on the plane

$$2x + y - z - 5 = 0$$

   that lies closest to the origin.

   (b) Find and classify the extreme values (if any) of the function

$$f(x,y) = y^2 + x^3.$$

End of Paper

25 points for each of the following problems

1. (a) Give the detail statement of the Central Limit Theorem (C.L.T) for the i.i.d. case.
(b) Suppose X is binomial b(100,1/3). Estimate $P\{X > 50\}$.
(c) Suppose $X_1, X_2, ..., X_{100}$ are i.i.d. Poisson P(λ) with $\lambda = 0.02$.
Approximte $P\{\sum_{i=1}^{100} X_i \geq 3\}$.

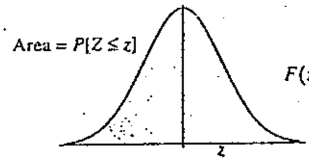
2. If $X_1, X_2, ..., X_n$ are i.i.d. random variables with density $f(x, \theta)$, where $f(x, \theta)$ satisfies the regular conditions. Answer the following questions:
(a) If $T(X_1, X_2, ..., X_n)$ is an unbiased estimator of θ, what is the Rao-Cramer lower bound for the variance of T.
(b) If X_i are i.i.d. normal $N(\mu, \sigma^2)$, where both μ and σ^2 are unknown. Is $\bar{X}$ the best estimator for μ? why?
(c) $S^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2/(n-1)$ is an unbiased estimator for σ^2. Does it attain the Rao-Cramer lower bound?

3. Suppose the six observations of X are 5,6,5,7,8,4 and the six observations of Y are 6,8,8,9,9,7. X and Y are independent. Answer the following questions:
(a) If X and Y come from the distributions F and G respectively. To test H_0: $F = G$ against H_1: $F \neq G$. Suppose we use Wilcoxon rank test (that is we give rank k to the k-th observation of the combined sample). Make your decision under the level 0.1.
(b) If X and Y come from the distributions $N(\mu_1, {\sigma_1}^2)$ and $N(\mu_2, {\sigma_2}^2)$ respectively. To test H_0: $\mu_1 = \mu_2$ against H_1: $\mu_1 \neq \mu_2$. Make your decision under the level 0.05.

4. (a) Suppose we have 1000 i.i.d. observations. Propose a simple method to test if these data come from $N(\mu, \sigma^2)$.
(b) If X is Weibull(α, β) with density $f(x; \alpha, \beta) = \beta\alpha^\beta x^{\beta-1}\exp\{-(\alpha x)^\beta\}$. Find the density of $Y = \log(X)$.
(c) Suppose we have 20 i.i.d. observations. Propose a simple method from (b) to test if these data come from Weibull(α, β).

Table A.3

Standard Normal c.d.f.



$$F(z) = P[Z \le z] = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990

25 points for each of the following problems

1. Let X_1 and X_2 be two independent uniform $U(0,1)$ random variables. Define $Y_1=\min\{X_1,X_2\}$, $Y_2=\max\{X_1,X_2\}$. Answer the following questions:
(a) Find the probability of $P\{|X_1-X_2|\le 0.3\}$
(b) Prove that if $Y=-2\log(X_1)$ then Y is distributed as $\chi^2(2)$
(c) Find the distribution of X_1/X_2
(d) Find the conditional density of Y_2 given $Y_1=0.5$.

2. (a) Prove that if X and Y are independent random variables then X^2 and Y^2 are also independent.
However the converse is not necessary true, the following is an example: Let (X,Y) have the joint density $f(x,y)=(1+xy)/4$ for $|x|\le 1$, $|y|\le 1$ and $f(x,y)=0$ otherwise.
(b) Prove that X and Y are not independent
(c) Prove that X^2 and Y^2 are independent.

3. Assume that X and Y have a joint density $f(x,y)=1$ for $|y|\le x, 0<x<1$ and $f(x,y)=0$ otherwise. Answer the following questions:
(a) Find the marginal p.d.f. of X and Y.
(b) Find $E(X|Y=y)$ and $E(Y|X=x)$.
(c) Find $Cov(X,Y)$.

4. A system consists of n components each of which will, independently work with probability p. The system will be able to operate effectively if at leat one-half of its compenents work. Answer the following questions:
(a) If $p=0.4$. Is a 5-component system more likely to operate effectively than a 3-component system?
(b) For what value of p is a 5-component system more likely to operate effectively than a 3-component system?
(c) Prove that if $p>0.5$ then a 7-component system is more likely to operate effectively than a 5-component system.

國立中山大學 89 學年度碩博士班招生考試試題

科　目：數值分析 應用數學系碩士班 乙　

Entrance Exam of Numerical Analysis for the Master Program of Scientific Computing

Five questions with the marks indicated.

I. (15) Define the vector and matrix norms,
$\|x\|_2 = \{\sum_{i=1}^{n} x_i^2\}^{\frac{1}{2}},\ \ x \in R^n,\ \ \|A\|_2 = \max_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2},\ \ A \in R^{(n\times n)}$.
Prove
(a) $\|x+y\|_2 \le \|x\|_2 + \|y\|_2$.
(b) $\|A\|_2 = \{\lambda_{max}(A^T A)\}^{\frac{1}{2}}$.

II. (20) To solve the linear algebraic equations $Ax = b$, we choose the Jacobi iteration method,

$$x^{(k+1)} = x^{(k)} - \alpha(Ax^{(k)} - b),\ \ k = 0, 1, ... \tag{1}$$

where $x^{(0)}$ is an initial value, and α is a factor to be decided. Suppose that matrix A is symmetric and positive definite, and its eigenvalues have a range: $0 < \lambda_{min} \le \lambda(A) \le \lambda_{max}$.
(a) Give a sufficient condition of choosing factor α for convergence of Jacobi method (1).
(b) Find the optimal factor α_{opt}.

III (15). To seek a root of a nonlinear equation $f(x) = 0$ by the Newton iteration method

$$x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})},\ \ k = 0, 1, ... \tag{2}$$

where $x^{(0)}$ is an initial value. Prove that the Newton iteration method (2) is convergent in a quadratic convergence rate, generally. Also give the conditions for such a high convergence rate.

IV. (20) Suppose that the data (x_i, y_i) are given with $x_i > 0,\ \ y_i > 0,\ \ n > 2$. The fitting curve is known as an exponential function $y = ce^{ax}$ with two constant a and c to be determined. Derive a computational formula to evaluate constants a and c, based on the least squares method.

V. (30) To solve the boundary problem of ordinary differential equations(ODE),
$-\frac{\partial^2 u}{\partial x^2} = f(x),\ \ x \in [0,1], u(0) = 0,\ \ u(1) = 0$. Denote $u_i = u(x_i)$ and $h_i = x_{i+1} - x_i$.
(a) Derive in details the difference equations, uniform and non-uniform, denoted by $Ax = b$.
(b) Derive their truncation errors.
(c) What are properties of matrix A in $Ax = b$ obtained.
(d) Give a good solution method for the difference equations derived, and count the number of flops (i.e., floating operations) needed.

Entrance Exam of Introduction to Computer for the Master Program of Scientific Computing

Six questions with the marks indicated.

I. (10) What are computer hardware and software?
To depict the basics of the computer hardware.

II. (10) Give the definition of algorithms.
What are binary encoding and computer programming?

III. (15) How to represent a real number in computer.
How many decimal significant digits in double precision?
Why?

IV. (15) Describe the way a computer performs really the integer operation:

$$78 + 150 = 228.$$

V. (20) What are basic arithmetic operations of computer?
What is rounding error of a computer?
Analyze the rounding errors resulting from $x\theta y$, where x and y are real, and θ is a basic computer operation.

VI. (30) To seek a root of $f(x)$, i.e., $f(x) = 0$, where

$$f(x) = P_n(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n$$

are polynomials of order n, and a_i are the known coefficients.
(a) Give an algorithm to compute $P_n(x)$ with less computer operations.
(b) Give the algorithm of binary search, or called the bisection method, to find the root of $P_n(x)$.
(c) Count the number of *flops*, the floating operations, needed in the above algorithms.
(d) Depict a flow chart (i.e, the flow boxes, or the logic of outlines) of computer programming.
(e) Write a computer program by any computer language.

Answer all 5 questions. Each worth 20 points.

1 Let A, B be two $n \times n$ matrices. Prove that A and B are row equivalent if and only if there exists an invertible matrix C such that $CA = B$.

2 Suppose A is a symmetric matrix, and that $\lambda_1 \neq \lambda_2$ are eigenvalues of A. Prove that any eigenvector $\vec{v}_1$ corresponding to λ_1 is orthogonal to any eigenvector $\vec{v}_2$ corresponding to λ_2.

3 Suppose W, V are subspace of R^n, $\dim(W) + \dim(V) = n$ and that every vector $\vec{w}$ of W is orthogonal to every vector $\vec{v}$ in V. Prove that for any vector $\vec{x}$ of R^n, there is a unique vector $\vec{w}$ of W and a unique vector $\vec{v}$ in V such that $\vec{x} = \vec{w} + \vec{v}$.

4. Suppose $B = ([2,3,1],[1,2,0],[2,0,3])$ and $B' = ([1,0,0],[0,1,0],[1,1,1])$ are ordered bases for R^3. Find the change-of-coordinates matrix from B to B'.

5. Define T on R^3 by $T([x_1, x_2, x_3]) = [x_1, -5x_1 + 3x_2 - 5x_3, 2x_2 - x_3]$. Find the eigenvalues and the corresponding eigenvectors of T.

國立中山大學八十九學年度碩博士班招生考試試題

科 目：高等微積分 應用數學系碩士班丙

ANSWER *all* 5 QUESTIONS, EACH OF WHICH CARRIES 20 POINTS.

1. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous with inverse $f^{-1} = f$.
   (a) Prove that there is at least one x in $\mathbb{R}$ with $f(x) = x$.
   (b) Prove that if f is non-decreasing then $f(x) = x, \forall x \in \mathbb{R}$.

2. (a) Prove that $f(x) = \sqrt{x}$ is continuous at $x = 1$ in the ϵ–δ language.
   (b) Is f uniformly continuous on the compact interval $[0, 1]$? Why?

3. Calculate $\int_A xy\, dxdy$ in each of the following cases.
   (a) A is the square in the plane $\mathbb{R}^2$ with vertices at points $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$.
   (b) A is the triangle in the plane $\mathbb{R}^2$ with vertices at points $(0, 0)$, $(0, 1)$ and $(1, 0)$.
   (c) A is the portion of the disk lying in the first quadrant in the plane $\mathbb{R}^2$ with center at the point $(0, 0)$ and of radius 1.

4. In the following, ln is the logarithmic function with base the natural constant $e = 2.718281827\ldots$.
   (a) Find the Taylor series representation of the function $\ln(1 + x)$.
   (b) Show that $\ln 2 = 1 - 1/2 + 1/3 - 1/4 + \ldots$.
   (c) Show that $\ln 2$ is not a rational number.

5. Define $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by setting $f(0, 0) = 0$, and
   $$f(x, y) = xy(x^2 - y^2)/(x^2 + y^2).$$
   (a) Show that the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ exist at $(0, 0)$.
   (b) Calculate $\partial f/\partial x$ and $\partial f/\partial y$ at $(x, y) \neq (0, 0)$.
   (c) Show that f is of class C^1 on $\mathbb{R}^2$; namely, all first partial derivatives of f exist and continuous at each point.
   (d) Show that $\partial^2 f/\partial x \partial y$ and $\partial^2 f/\partial y \partial x$ exist at $(0, 0)$, but are not equal there.

End of Paper

In the following, $\mathbb{C}$ is the set of complex numbers, and $\mathbb{R}$ is the set of real numbers.

Problem 1. Describe the set

$$L = \{z \in \mathbb{C} : \frac{|z-i|}{|z-1|} = \sqrt{2}\}$$

explicitly and geometrically. (10 %)

Problem 2. Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be defined by

$$f(x+iy) = (x^3 - x^2 - 4y) + i(3x^2y - 2xy + 4x) \quad \text{for } x,\ y \in \mathbb{R}.$$

(i) Prove that f is differentiable at every $r \in \mathbb{R}$, and find $f'(r)$. (10 %)

(ii) Is f analytic at the origin ? Prove your answer. (10 %)

Problem 3. Let $\gamma(\theta) = \rho e^{-i\theta}$, where $\rho > 1$ and $0 \le \theta \le 2\pi$. Let

$$f(z) = \frac{\sin \pi z}{(z-1)^8} \qquad \text{for all } z \in \mathbb{C} - \{1\}.$$

(i) Find the Laurent series expansion of f at 1. (10 %)

(ii) Find the residue of f at 1, and evaluate the integral $\int_\gamma f(z)\,dz$. (10 %)

Problem 4. Let f be a function analytic on the whole complex plane, and let V be the imaginary part of f. Assume that there is a constant $m \in \mathbb{R}$ such that $V(z) \ge m$ for all $z \in \mathbb{C}$.

(i) If $g(z) = e^{if(z)}$ for all $z \in \mathbb{C}$, prove that g is constant on $\mathbb{C}$. (10 %)

(ii) Prove that f is constant on $\mathbb{C}$. (10 %)

Problem 5. Let f be a function analytic on $\mathbb{C}$. If $|f(z)| \le |z|^5$ for all $z \in \mathbb{C}$, prove that f is a polynomial. (10 %)

Problem 6. Let $S = \{z \in \mathbb{C} : |6z-1| = 1\}$, and let f be the Möbius transformation with $f(1) = 1$, $f(0) = 2$ and $f(-1) = \frac{5}{3}$.

(i) Write f in the form $f(z) = \frac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{C}$. (10 %)

(ii) Describe the set $f(S)$ explicitly and geometrically. (10 %)