國立中山大學九十一學年度碩士班招生考試試題

科目：基礎數學A【應數系碩士班甲組】

共七題，滿分100分。答題時，每題都必須寫下題號與步驟。

1. Let $A_n = (a_{ij})$ be an $n \times n$ matrix where $a_{ij} = \alpha$ if $i = j$ and 1 otherwise. For example

$$A_4 = \begin{pmatrix} \alpha & 1 & 1 & 1 \\ 1 & \alpha & 1 & 1 \\ 1 & 1 & \alpha & 1 \\ 1 & 1 & 1 & \alpha \end{pmatrix}.$$

   (a) Find the determinant, inverse, and eigenvalues of A_4. (20分)

   (b) Find the inverse of A_n and determine if A_n is a strictly positive definite matrix. (20分)

2. Calculate $\int \frac{3 + x + 2x^2}{1 + x + x^2 + x^3}\, dx$. (10分)

3. Use two iterates of Newton method with initial value 1 to approximate $\sqrt[3]{2}$. (10分)

4. Find the area inside the circle $r = 5 \sin \theta$ and outside the limacon $r = 2 + \sin \theta$. (10分)

5. Find the area bounded by the two curves $y = x^2 + 3x + 5$ and $y = -x^2 + 5x + 9$ for x between -1 and 4. (10分)

6. Let $f(x, y) = 2x^3 - 24xy + 16y^3$. Determine the nature of the critical points of f. (10分)

7. Find the volume of the tetrahedron formed by the planes $x = 0, y = 0, z = 0$, and $x + (y/2) + (z/4) = 1$. (10分)

～全卷完～

25 points for each of the following problems

(Calculators are permitted)

1. Let X_1, X_2, X_3 be i.i.d exponential r.v's with probability density function (p.d.f) $f(x,\theta) = (1/\theta)exp(-x/\theta)$, $x \geq 0$, $\theta > 0$.
(a) Prove that $X_1 + X_2$ and X_1/X_2 are independent. (8 pts)
(b) Find the p.d.f of the random variable $Z = X_1/(X_1 + X_2)$. (8 pts)
(c) Suppose $X_{(1)} \leq X_{(2)} \leq X_{(3)}$ are the order statistics and let $W_1 = 3X_{(1)}$, $W_2 = 2(X_{(2)} - X_{(1)})$, $W_3 = (X_{(3)} - X_{(2)})$. Prove that W_1, W_2 and W_3 are i.i.d random variables. (9 pts)

2. Let $X_1, X_2, ..., X_n$ be a sample drawn from uniform distribution $U[0,\theta]$ with p.d.f $f_\theta(x) = 1/\theta$ if $0 \leq x \leq \theta$, and 0 otherwise. Answer the following questions: (5 pts each)
(a) Compute the Cramer-Rao lower bound for the variance of an unbiased estimator of θ.
(b) If $T = ((n+1)/n)Max(X_1, X_2, ..., X_n)$, show that T is an unbiased estimator of θ.
(c) Prove that $Max(X_1, X_2, ..., X_n)$ is a complete sufficient statistic .
(d) Compute the variance of T constructed in (b).
(e) The variance of T is smaller than that of the value in (a), does it contradict the Cramer-Rao Theorem? why?

3. Answer the following three questions.
(a) Let $X_1, X_2, ..., X_n$ be a sample with p.d.f $f(x,\theta)$. Give a definition of the "MostPowerful(MP)" size α test for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$. Explain how to obtain such a test. (8 pts)
(b) Let X be an observation in the interval (0,1) with p.d.f $f(x)$. Find an MP size $\alpha = 0.05$ test of H_0: $f(x) = 2x$ for $0 < x < 1/2$ and $f(x) = 2 - 2x$ for $1/2 \leq x \leq 1$ against H_1: $f(x) = 1$ for $0 \leq x \leq 1$. (8 pts)
(c) Let X_1, X_2, X_3, X_4, X_5 be 5 samples from p.d.f $f(x,\theta) = \theta/x^2$ if $0 < \theta \leq x < \infty$. Find an Mp size $\alpha = 0.05$ test of $H_0 : \theta = 1$ against $H_1 : \theta = 2$. (9 pts)

4. Let $X_1, X_2, ..., X_n$ and $Y_1, Y_2, ..., Y_m$ be random samples drawn from X and Y respectively where X has distribution function F and Y has distribution function G. The testing hypothesis is $H_0 : F = G$ against $H_1 : F \neq G$. Let $Z_{(1)} \leq Z_{(2)} \leq ... \leq Z_{(n+m)}$ be the order statistics of the combined samples. Define $W = \sum_{k=1}^{n+m} kI_k$ where $I_k = 1$ if $Z_{(k)}$ is an X and $I_k = 0$ if $Z_{(k)}$ is an Y.
(a) Compute the mean and variance of the random variable W under H_0. (10 pts)
(b) Suppose in an experimental project, Hospital A applys its treatment, say treatment A, on 10 cancer patients and they survive 3.1, 2.8, 3.2, 5.1, 4.2, 2.1, 1.5, 3.6, 4.3, 4.7 years respectively. At the same time Hospital B applys its treatment, say treatment B, on 12 cancer patients and they survive 1.2, 2.4, 3.5, 2.8, 2.3, 4.4, 8,2, 2.5, 1.1, 2.7, 1.8, 0.6 years respectively. Use the result in part (a) to make a judgement about these two treatments. Type I error is 0.05. (15 pts)

國立中山大學九十一學年度碩士班招生考試試題

科目：機 率 論【應數系碩士班甲組】

(第1-4 題每題16 分，第5-6題18 分)

1. 一氣體分子在平衡狀態下之速度爲一隨機變數X, 其機率密度函數 (p.d.f.)爲

$$f(x)=\begin{cases} k\,x^2\,e^{-bx^2}, & x>0 \\ 0, & x\le 0, \end{cases}$$

其中k 爲標準化常數, $b>0$ 與氣體溫度及分子質量有關.
(i)試求k 之值. (ii) 試求動能$E=m\,X^2/2$之p.d.f.

2. 設Y 有$Be(\alpha,\beta)$分佈, 而X 表投擲一出現正面之機率爲Y 之銅板n 次所得之正面數.

   (i) 試求 X之邊際p.d.f.
   (ii) 試求給定 $X=x$, Y 之條件分佈.

3. 設$U_1,U_2,\ldots,$ 爲i.i.d. U(0,1) 隨機變數數列.

   (i) 令$Y=\min\{U_1,1-U_1\}$, 試求Y 之分佈.
   (ii) 令N 爲第一個大於或等於2 的正整數n, 使得$U_n>U_{n-1}$.
   i.e. $N=\min\{n|n\ge 2, U_n>U_{n-1}\}$. 對$0\le u\le 1$:
   (a). 試證$P(U_1\le u$ 且 $N=n)=\frac{u^{n-1}}{(n-1)!}-\frac{u^n}{n!}$, $n\ge 2$.
   (b). 試求 $P(U_1\le u$ 且n爲偶數).

4. 對一數列之r.v.'s $\{X_n,n\ge 1\}$ 及一r.v. X, 當$n\to\infty$,

   (i) 試定義$\{X_n,n\ge 1\}$ 分佈收斂(convergence in distribution)至X.
   (ii) 設一隨機變數被稱爲具有參數$a,\,b\,(a>0,b>0)$的Pareto 分佈, 若其p.d.f. 爲

$$f(x|a,b)=\frac{a}{b(1+x/b)^{a+1}},\ x>0.$$

   令$X_1,\ldots,X_n$爲具有參數$a,\,b\,(a>0,b>0)$的Pareto 分佈之i.i.d.隨機樣本. 試求隨機變數$U_n=n\min(X_1,\ldots,X_n)$之極限分佈.

5. 對一數列之r.v.'s $\{X_n,n\ge 1\}$ 及一r.v. X ,且令$\overline{X}_n=\sum_{i=1}^n X_i/n$.

   (i) 試定義當$n\to\infty$, $\{X_n,n\ge 1\}$機率收斂(convergence in probability)至X, 即$X_n \xrightarrow[n\to\infty]{P} X$.
   (ii) 設$\{X_n,n\ge 1\}$爲i.i.d., 且設$\mu=E(X_1)$, $Var(X_1)=\sigma^2$ 皆存在. 試證$\overline{X}_n \xrightarrow[n\to\infty]{P} \mu$.
   (iii) 設$\{X_n,n\ge 1\}$爲獨立. X_n 分佈$(n=1,2,\ldots,)$之p.d.f. 爲
   $P(X_n=\mu+n^\alpha)=P(X_n=\mu-n^\alpha)=1/2,\ n=1,2,\ldots,$
   試證若$0<\alpha<1/2$, $\overline{X}_n \xrightarrow[n\to\infty]{P} \mu$.

6. 一不平衡的骰子其出現點數k, $k=1,2,\ldots,6$ 之機率與k成正比. 一遊戲每投擲一次骰子要先付出4百元, 若投出的數字爲k, 則可獲得k百元.

   (i) 設獨立地投擲骰子100次, 令點數k總共出現之次數爲X_k, 試說明$X=(X_1,X_2,\ldots,X_6)$之分佈爲何?
   (ii) 試求$E(\prod_{k=1}^6 s_k^{X_k})$, $s_k>0$, $E(X_k)$, 及$Var(X_k)$, $k=1,\ldots,6$.
   (iii) 當遊戲進行到投擲骰子100次後, 其淨所得爲正的機率爲何?

Linear Algebra

注意：每個題目需證明或說明清楚。

Let $\mathbf{R}$ be the set of all real numbers and $M_{m\times n}(\mathbf{R})$ be the set of all m×n matrices over $\mathbf{R}$.

1. Let $S=\{(x,y,z): x+y+z=0\}\subseteq \mathbf{R}^3$.
   (a) Show that S is a subspace of $\mathbf{R}^3$ and find a basis of S. (10%)
   (b) Suppose P is a projection operator on $\mathbf{R}^3$, i.e. $P^2=P$, the image of P is S, and $P(1,1,1)=(0,0,0)$. Find $P(x,y,z)$. (10%)

2. Prove or disprove: suppose V is a finite-dimensional vector space over $\mathbf{R}$ and T is a linear operator on V. Then T is one-to-one if and only if T is onto. (10%)

3. Let AX=B be a linear system with $A\in M_{m\times n}(\mathbf{R})$, $X\in M_{n\times 1}(\mathbf{R})$, and $B\in M_{m\times 1}(\mathbf{R})$.
   Show that the linear system AX=B has a unique solution if and only if the columns of A are linearly independent and B is a linear combination of the columns of A. (10%)

4. Let $\alpha=\{(1,0,0),(1,1,0),(1,1,1)\}$ be an ordered basis of $\mathbf{R}^3$ and define $T: \mathbf{R}^3\to\mathbf{R}^3$ by $T(x,y,z)=(x,x+y,x+y+z)$.
   (a) Show that T is a linear transformation. (4%)
   (b) Find the matrix representation A of T with respect to α. (6%)
   (c) Find an orthogonal matrix Q and an upper-triangular matrix U satisfying A=QU. (8%)
   (d) Find the characteristic polynomial and the minimal polynomial of T. (8%)
   (e) Find the Jordan canonical form of T. (6%)

5. Let a nonsingular matrix $A=[a_{ij}]\in M_{n\times n}(\mathbf{R})$, A_{ij} be the cofactor of a_{ij} in A, and the cofactor matrix $B=[A_{ij}]\in M_{n\times n}(\mathbf{R})$. Prove that A is diagonalizable if and only if B is diagonalizable, too. (12%)

6. Suppose $A,B\in M_{m\times n}(\mathbf{R})$, a nonsingular $Q\in M_{m\times m}(\mathbf{R})$, and A=QB.
   (a) Show that A and B have the same row space. (6%)
   (b) Prove or disprove: if B and C are reduced row echelon form and there exists a nonsingular $P\in M_{m\times m}(\mathbf{R})$ such that A=PC, then B=C. (10%)

Entrance Exam of Calculus for the Master Program of Scientific Computing

Full marks are 100; the marks are indicated within questions.

I. (10) Give the definitions of limit of functions and continuity of functions respectively, and state their differences.

II. (15) Find the sum of the series,

$$x+\frac{x^3}{3}+\frac{x^5}{5}+...,\quad -1<x<1.$$

III. (15) Prove the limit

$$\lim_{x\to x_0}\sqrt{x}=\sqrt{x_0}$$

by the $\epsilon-\delta$ definition.

IV. (15) Find the sum of the limit

$$\lim_{n\to\infty}\left(\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+...+\frac{n}{n^2+n^2}\right).$$

V. (15) Evaluate the integral on three dimensions,

$$I=\int\int\int_{\Omega}(x^2+y^2)dv,$$

where the integration region Ω is surrounded by the rotating parabola $z=x^2+y^2$ and the plane $z=1$.

VI. (15) Expand $sinx$ into polynomials with two terms by Taylor's series, compute the approximate value of $sin(0.1)$ (i.e. $sinx$ when $x=0.1$), and estimate the absolute and relative errors.

VII. (15) Prove that the functions

$$r^{i+\frac{1}{2}}cos(i+\frac{1}{2})\theta$$

satisfy the Laplace equation,

$$\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,$$

where (r,θ) are the polar coordinates.

Twenty points for each problem. Please write down all the detail of your computation and answers.

1. Write a program to evaluate the polynomial

$$f(x) = \sum_{i=0}^{n} a_i x^i$$

and its derivative at $x = c$ using least arithmetic operation. Compute the number of arithmetic operation needed in your program.

2. List five methods that you know to find a root of a given nonlinear equation. State the advantage, disadvantage and the order of convergence of each method.

3. Use both the Lagrange formula and the Newton divided difference formula to compute the polynomial $p(x)$ interpolating the following data

x	-2	-1	0	2
$f(x)$	-19	-2	1	13

What is the error $f(x) - p(x)$ at $x = c$?

4. Use polynomial interpolation to prove the three-point midpoint formula for second derivative

$$f''(c) \approx \frac{1}{h^2}[f(c+h) - 2f(c) + f(c-h)],$$

and use Taylor's Theorem to compute its error formula. Is this a stable method?

5. Assume the Gaussian elimination on $n \times n$ matrix A needs no row exchange. Write a program to compute the LU (triangular) factorization of A using the least memory. How much storage is needed to run your program?

國立中山大學九十一學年度碩士班招生考試試題

科目：高等微積分【應數系碩士班丙組】

ANSWER *all* 5 QUESTIONS, EACH OF WHICH CARRIES 20 POINTS.

1. (a) Use the method of *Lagrange's multipliers* to find the least distance of the point $(1,2,3)$ to the plane
$$4x + 5y + 6z = 7.$$
   (b) Find and classify the extreme values (if any) of the function
$$f(x,y) = y^2 + x^3.$$

2. (a) Prove that $f(x) = 1/\sqrt{1-x}$ is continuous from the left at $x = 1$ in the ϵ–δ language.
   (b) Is f uniformly continuous on the open interval $(0,1)$? Why?

3. (a) If $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is of class C^1, show that f is not one-to-one.
   (b) If $f : \mathbb{R} \longrightarrow \mathbb{R}^2$ is of class C^1, show that f does not carry $\mathbb{R}$ onto $\mathbb{R}^2$. In fact, show that $f(\mathbb{R})$ contains no open subset of $\mathbb{R}^2$.

4. (a) Prove that every convex function $f : [0,1] \longrightarrow \mathbb{R}$ attains its maximum value at either 0 or 1.
   (b) Prove that the unit interval $[0,1]$ is a compact set. But you *cannot* just quote the fact that bounded and closed sets in $\mathbb{R}$ are compact.

5. Let $I = [0,1]$ and $Q = I \times I$. Define $f : Q \longrightarrow \mathbb{R}$ by
$$f(x,y) = \begin{cases} \frac{1}{p} & \text{if } y \text{ is rational and } x = \frac{q}{p}, p, q \in \mathbb{N} \text{ such that} \\ & \text{the greatest common factor } (p,q) \text{ of } p \text{ and } q \text{ is } 1, \\ 0 & \text{otherwise.} \end{cases}$$
   (a) Is f integrable over Q? If yes, compute $\int_Q f$.
   (b) Compute the upper integral $\underline{\int}_{y\in I} f(x,y)$ and the lower integral $\overline{\int}_{y\in I} f(x,y)$.
   (c) Show that $\int_{y\in I} f(x,y)$ exists for x in $I - D$, where D is a set of measure zero in I.
   (d) Verify Fubini's theorem for $\int_Q f$.

(End of Paper)

國立中山大學九十一學年度碩士班招生考試試題

科目：複變函數論【應數系碩士班丙組】

In the following, $\mathbf{C}$ is the set of complex numbers, $\mathbb{R}$ is the set of real numbers, and i is a square root of -1.

Problem 1 Let $D = \{z \in \mathbf{C} : |z| < \frac{\pi}{2}\}$, and let $f(z) = |z| \cdot |\tan z|$ for $z \in D$.

(i) Prove that f is differentiable at 0. (10 %)

(ii) Prove that f is not analytic at 0. (10 %)

Problem 2 Let $\gamma(\theta) = 8e^{i\theta}$, $0 \le \theta \le 2\pi$. Evaluate : $\int_\gamma \frac{\sin z}{1 - \cos z}\, dz$ (15 %)

Problem 3 Let $h(\theta) = \frac{1}{(2 + \cos\theta)^2}$ for $0 \le \theta \le 2\pi$. Evaluate : $\int_0^{2\pi} h(\theta)\, d\theta$ (20 %)

Problem 4 Let $f : \mathbf{C} \longrightarrow \mathbf{C}$ be an analytic function. Assume that there is a real number $p > 0$ such that

$$|f(z)| \le |z|^p \quad \text{for all } z \in \mathbf{C}.$$

Prove that f is a polynomial. (15 %)

Problem 5 Let Ω be a nonempty open and connected subset of the complex plane $\mathbf{C}$, let $f : \Omega \longrightarrow \mathbf{C}$ be an analytic function, and let $u(z)$ be the real part of $f(z)$ for every $z \in \Omega$. Assume that there is a $z_0 \in \Omega$ such that

$$u(z_0) = \max_{z \in \Omega} u(z).$$

Prove that f is constant on Ω. (15 %)

Problem 6 Let $S = \{z \in \mathbf{C} : |z| = 1\}$, and let f be the Möbius transformation such that $f(S) = S$, $f(1) = i$ and $f(2) = \frac{1+i}{2}$. Write f in the form

$$f(z) = \frac{az + b}{cz + d},$$

where $a, b, c, d \in \mathbf{C}$. (15 %)