國立中山大學95學年度碩士班招生考試試題

科目：基礎數學A【應數系碩士班甲組】

共十題，每題10分。答題時，每題都必須寫下題號與詳細步驟。

1. Find the determinant of the matrix **A** where

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 4 & 2 & 2 & 1 \\ 4 & 2 & 2 & 3 \\ 3 & 1 & 4 & 1 \end{bmatrix}.$$

2. For the matrix **A** below, the identity matrices are each 3×3. Find the inverse of **A**.

$$\mathbf{A} = \begin{bmatrix} 3\mathbf{I} & 2\mathbf{I} \\ -\mathbf{I} & 4\mathbf{I} \end{bmatrix}.$$

3. Show that the largest characteristic root of a correlation matrix is less than or equal to n, the size of the matrix.

4. Let $\mathbf{a}$ be a $n \times 1$ vector such that $\mathbf{a}'\mathbf{a} = 1$. Find the eigenvectors and eigenvalues of $\mathbf{I} - 2\mathbf{a}\mathbf{a}'$.

5. If $\mathbf{A} - \mathbf{B}$ is non-negative, prove or disprove that $\mathbf{A}^2 - \mathbf{B}^2$ is non-negative.

6. Let $f(x)$ be a convex function on $D \subset \mathbb{R}$. Show that $\exp[f(x)]$ is also a convex on D.

7. Let the function $f(x)$ be defined as

$$f(x) = \begin{cases} x^3 - 2x, & x \geq 1, \\ ax^2 - bx + 1, & x < 1. \end{cases}$$

For what values of a and b does $f(x)$ have a continuous derivative?

8. Show that the sequence $\{a_n\}_{n=1}^{\infty}$ converges, and find its limit, where $a_1 = 1$ and

$$a_{n+1} = (2 + a_n)^{1/2}, \qquad n = 1, 2, \ldots.$$

9. Determine whether the following integrals is convergent or divergent:

$$\int_0^{\infty} \frac{dx}{\sqrt{1 + x^3}}.$$

10. Determine the stationary points of the following functions and check for local minima and maxima: $f = 2\alpha x_1^2 - x_1 x_2 + x_2^2 + x_1 - x_2 + 1$, where α is a scalar. Can α be chosen so that the stationary point is (i) a point of local minimum; (ii) a point of local maximum; (iii)a saddle point?

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(1) Let X be a random variable with moment generating function $M(t)$ and set $K(t) = \log M(t)$ for those t's for which $M(t)$ exists. Suppose that $E(X) = \mu$ and $Var(X) = \sigma^2$ are both finite. Show that $\frac{d^2}{dt^2}K(t) = \sigma^2$.(10pts)

(2) Let X be a random variable such that

$$E(X^{2k}) = \frac{(2k)!}{k!}, \quad k = 0, 1, 2, \cdots.$$

Find the moment generating function of X and then deduce the distribution of X.(15pts)

(3) Let X_1 and X_2 be iid random variables with density $f(x) = 2x$, $x \in (0,1)$ and let $Y = X_1 + X_2$. Find the density function of Y.(15pts)

(4) Suppose that $X_1, X_2, \cdots, X_n$ are iid random variables with the density function

$$f(x|\alpha) = \frac{\Gamma(3\alpha)}{\Gamma(\alpha)\Gamma(2\alpha)} x^{\alpha-1}(1-x)^{2\alpha-1} \quad \text{for} \quad x \in [0,1],$$

where $\alpha > 0$, is a parameter to be estimated. Find the Cramer-Rao lower bound on the variance of any unbiased estimate of α.(15pts)

(5) Let $X_1, X_2, \cdots, X_n$ be iid random variables with pdf

$$f(x) = \vartheta x^{\vartheta-1} \text{ for } x \in [0,1].$$

Let $Y_j = -2\vartheta \log X_j, j = 1, 2, \cdots, n$. Show that $T_n(\vartheta) = -2\vartheta \sum_{j=1}^{n} Y_j$ is a $\chi^2(2n)$ random variable. Based on this result find a $100(1-\alpha)\%$ confidence interval for ϑ.(15pts)

(6) Assume that $X_1, X_2, \cdots, X_n$ are iid Poisson(λ) random variables. Let

$$Z_i = \begin{cases} 1 & \text{if } X_i = 0 \\ 0 & \text{if } X_i \neq 0 \end{cases}$$

$i = 1, 2, \cdots, n$. Show that $\frac{\sum_{i=1}^{n} Z_i}{n}$ is an unbiased estimator of $e^{-\lambda}$. Find an UMVUE (uniformly minimum variance unbiased estimate) of $e^{-\lambda}$.(15pts)

(7) Suppose that $X_1, X_2, \cdots, X_n$ are iid $N(\mu, \sigma^2)$ random variables, assuming the mean μ is known. Derive the generalized likelihood ratio test for

$$H_0 : \sigma^2 = \sigma_0^2 \quad \text{versus} \quad H_a : \sigma^2 \neq \sigma_0^2.$$

Also write down the power function of the test.(15pts)

第 1-5 題，每題20 分。

1. 設r.v.X有一連續的p.d.f.f。
   (i) 試寫出$Y = X^2$之p.d.f. 之形式。
   (ii) 若X有標準常態分佈 $N(0,1)$，試求$Y = X^2$ 之p.d.f.，說明其屬於哪一個分佈族(distribution family) 並求對應之特徵函數
   $$\psi(t) = E(e^{itX}) = \int_{-\infty}^{\infty} e^{itx} dF(x),\ t \in R.$$

2. (i) 設X_1與X_2獨立且分別有$\mathcal{B}e(r_1, s_1)$及$\mathcal{B}e(r_2, s_2)$分佈。令$Y_1 = X_1$, $Y_2 = X_2(1 - X_1)$，試求 Y_1, Y_2之聯合p.d.f.。
   (ii) 設X_1, X_2為i.i.d.之$\mathcal{N}(0,1)$ r.v.'s，試證 $X_1 - X_2$與 $X_1 + X_2$獨立。

3. (i) 設X, Y之聯合p.d.f.為$f(x,y) = k(k-1)(y-x)^{k-2}, 0 < x \le y < 1, k \ge 2$為一整數。試求$E(X|Y)$，且利用此一結果求$E(E(X|Y))$。
   (ii) 設X有$\mathcal{P}(\lambda)$分佈，且給定$X = k$, Y有$\mathcal{B}(k,p)$分佈。試證Y與$X - Y$獨立，並求給定$Y = y$, X之條件分佈。

4. 設有一射手在打靶練習時，其射擊點之分佈大致如(X, Y)，其中X, Y 為二獨立之隨機變數，且以$\mathcal{N}(0, \sigma^2)$ 為其共同分佈。
   (i) 試估計此射手所射擊之點，會落在以靶正中央為圓心，落在圓內之機率為0.5之半徑$r, r > 0$ 為何？
   (ii) 試推估此射手射擊點與靶心的平均距離。

5. 設$X_1, X_2, \cdots, X_n$ 為i.i.d.之隨機變數，且具有共同之連續d.f. F。令$X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$ 為其對應之順序統計量。
   (i) 試證$F(X_1)$ 有$\mathcal{U}(0,1)$分佈。
   (ii) 令$Y_i = F(X_{(i)}), i = 1, \cdots, n$，試求$Y_1, Y_n$ 之聯合分佈，及$P(Y_n - Y_1 > t), t \in (0,1)$。
   (iii) 試證上述之Y_n 機率收斂至1, 當$n \to \infty$。

Numerical Analysis Entrance Exam. for the Master Program 2006

If you think that a problem has been stated incorrectly, indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

1. (a) **[10 points]** Show that the Newton forward divided-difference polynomials

$$P(x) = 3 - 2(x+1) + 0(x+1)(x) + (x+1)(x)(x-1)$$

and

$$Q(x) = -1 + 4(x+2) - 3(x+2)(x+1) + (x+2)(x+1)(x)$$

both interpolate the data

x	-2	-1	0	1	2
$f(x)$	-1	3	1	-1	3

(b) **[10 points]** Why does part(a) not violate the uniqueness property of interpolating polynomials ?

2. (a) **[15 points]** Describe the method of Steepest Descent.

(b) **[15 points]** Use the method of Steepest Descent to approximate minima to within 0.005 for the function

$$g(x_1, x_2) = 100(x_1^2 - x_2)^2 + (1 - x_1)^2$$

3. (a) **[10 points]** Let $f : \mathbb{R} \to \mathbb{R}$ be a real value function on real numbers. Define the condition number of f and explain the meaning of condition number.

(b) **[10 points]** Let $y_n = \int_0^1 \frac{t^n}{t+5}\, dt$. Prove $y_k = -5y_{k-1} + \frac{1}{k}$, $k = 1, 2, \ldots$

(c) **[10 points]** We propose to compute y_n recursively by relating y_k to y_{k-1}. Therefore we have a problem $f_n : \mathbb{R} \to \mathbb{R}$, and $y_n = f_n(y_0)$. Prove that the condition number of f_n goes to infinity as n goes to infinity.

4. **[20 points]** For any two polynomials f and g on $\mathbb{R}$, we define $(f, g) = \int_{\mathbb{R}} f(x)g(x)\, dx$. We call that f and g are orthogonal if $(f, g) = 0$. Let $\{\pi_k(x) : k = 1, 2, \ldots, \pi_k$ are monic polynomials.$\}$ be a set of orthogonal polynomials. Prove that three consecutive orthogonal polynomials are linearly relatived.

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國立中山大學95學年度碩士班招生考試試題

科目：微積分【應數系碩士班乙組】

1. (15%) 假設 $0 \le x, y \le 1$, 試證 $|x-y| \le |e^x - e^y| \le 3|x-y|$.

2. (15%) 有一圓錐形容器, 高為 15 公尺, 頂半徑為 5 公尺, 今以每分鐘 2 立方公尺的速度注水入容器中, 問水高為 7 公尺時, 水面上升之速度為多少?

3. (15%) 今有一圓柱形的罐頭, 高度為y、底半徑為x。請問在體積固定情況下, 此罐頭有最小表面積時,$x : y$ 應為多少?

4. (20%) 請繪出 $y = \sqrt{x^2+1} - x + 1$ 之圖形, 並繪出遞增、遞減、凹凸與漸近線情形。

5. (15%) 求函數 $f(x,y,z) = x^4 + y^4 + z^4$ 在曲面 $x^2 + y^2 + z^2 = 1$ 上之最大值與最小值。

6. (20%) 令S為曲面$x^2 + y^2 = z^2$及$z = 1$所圍成之區域, 求S之體積。

Linear Algebra (注意：每題必需證明或說明清楚，只填答案不計分。)

Let $\mathbb{R}$ be the set of all real numbers, $\mathbb{C}$ be the set of all complex numbers, and $M_{m\times n}(F)$ be the set of all m×n matrices over a field F.

1. Let T be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ with T(1,1,1)=(1,2,3), T(1,−1,1)=(1,0,1) and T(3,−1,−2)=(3,2,0). Find $T(x,y,z)$, the characteristic polynomial, and the kernel of T. (15%)

2. Let $L(\mathbb{R}^3,\mathbb{R}^2)$ be the set of all linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^2$. Define $(f+g)(v)=f(v)+g(v)$ and $(rf)(v)=r(f(v))$ for $f,g\in L(\mathbb{R}^3,\mathbb{R}^2)$, $v\in\mathbb{R}^3$, $r\in\mathbb{R}$. Find a basis and the dimension of $L(\mathbb{R}^3,\mathbb{R}^2)$. (15%)

3. Suppose $A\in M_{5\times 5}(\mathbb{R})$. Prove that there exist an orthogonal matrix Q and a matrix $B=[b_{ij}]$ with $b_{ij}=0$ for $i+j\geq 7$ such that A=QB. (15%)

4. Prove that if $A\in M_{n\times n}(\mathbb{C})$ then there exist a diagonal matrix D and a nonsingular matrix P such that $(A+PDP^{-1})^n=0$ is the n×n zero matrix. (15%)

5. Determine each following statement either is true or false. If true, prove it; if false, give a counterexample. (8%×5)
   (a) Suppose $A, B\in M_{m\times n}(\mathbb{R})$, $x\in M_{n\times 1}(\mathbb{R})$, and $b\in M_{m\times 1}(\mathbb{R})$. If the linear systems $Ax=b$ and $Bx=b$ have the same solution set, then A and B are similar.
   (b) Suppose $A, B\in M_{n\times n}(\mathbb{R})$. If there exists a uniquely nonsingular matrix C such that A=BC then A and B are nonsingular, too.
   (c) Suppose S and T are linear operators on a finite dimensional vector space V, α and β are ordered bases for V. If $[S]_\alpha=[T]_\beta$ then S=T where $[S]_\alpha=[T]_\beta$ are the matrix representations of S and T with respect to α and β, respectively.
   (d) Let α be a basis for a finite dimensional vector space V and $T(\alpha)=\{T(v): v\in\alpha\}$. If T is a linear operator on V then T is onto if and only if $T(\alpha)$ is a basis for V.
   (e) Let T be the linear operator on $\mathbb{R}^3$, $f(x)=(x+1)(x+2)^2$ and $g(x)=(x+1)^2(x+2)$. If $f(T)=g(T)=0$ is the zero transformation, then T is diagonalizable.

Entrance Exam of Advanced Calculus for Master Programs of Departments of Applied Mathematics

Full marks are 100; Each question is 20 marks.

1. Find the limit of the following sequence $\{x_N\}$,

$$\sqrt{2}, \sqrt{2+\sqrt{2}}, \ldots, \sqrt{2+\sqrt{2+\sqrt{2+\ldots}}} \text{ involving } n \text{ sqare roots }, \ldots$$

2. Find the limit

$$\lim_{h\to 0} \frac{1}{h} \int_x^{x+h} \frac{du}{u+\sqrt{u^2+1}}.$$

3. Find definite integral

$$\int\int_S (x^2+y^2)dxdy,$$

where S is a disk: $x^2+y^2 \le R^2, \quad R>0$.

4. Find indefinite integral

$$\int \frac{xe^x}{\sqrt{e^x-1}}dx.$$

5. Prove that the following functions

$$r^n \cos n\theta, \quad r^n \sin n\theta, \quad n = 1, 2, \ldots$$

satisfy the Laplace equation,

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

where (r, θ) are the polar coordinates, $x = r\cos\theta$ and $y = r\sin\theta$.

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科目：複變函數論【應數系碩士班丙組】

In the following, $\mathbf{C}$ is the set of complex numbers, $\mathbb{R}$ is the set of real numbers, and i is a square root of -1. For any $z \in \mathbf{C}$, $\operatorname{Re} z$ denotes the real part of z, while $\operatorname{Im} z$ denotes the imaginary part of z.

Problem 1 Prove or disprove the following statements. (32 %)

(i) Let Ω be a nonempty open subset of $\mathbf{C}$, and let $f : \Omega \longrightarrow \mathbf{C}$ be a function. If f is differentiable at a point $z_0 \in \Omega$, then f is analytic at z_0.

(ii) Let Ω be a nonempty open and connected subset of $\mathbf{C}$. For a given analytic function $f : \Omega \longrightarrow \mathbf{C}$, let $u(z) = \operatorname{Re} f(z)$ and $v(z) = \operatorname{Im} f(z)$ for $z \in \Omega$. If $\{u(z)\}^2 - \{v(z)\}^2 = 1$ for all $z \in \Omega$, then f is a constant function on Ω.

(iii) There is a real number $r > 0$ such that $|\sin z| \le r$ for all $z \in \mathbf{C}$.

(iv) If $r > 0$, $\rho > 0$ and $\alpha \in \mathbf{C}$, then $\{z \in \mathbf{C} : |z - \alpha| < r\} \cap \{e^z : |z| > \rho\}$ contains an infinite number of points.

Problem 2 Let $\gamma(\theta) = 2\cos\theta + i\sin\theta$ for $0 \le \theta \le 2\pi$. Evaluate the line integrals :

(i) $\displaystyle\int_\gamma \frac{1}{2z^2 + 6z + 1}\, dz$ (8 %) (ii) $\displaystyle\int_\gamma \frac{\sin(\pi z)}{(z^2 + 4z + 3)^3}\, dz$ (10 %)

Problem 3 Let $f(x) = \dfrac{x^2}{x^4 + 5x^2 + 6}$ for $x \in \mathbb{R}$. Evaluate : $\displaystyle\int_0^\infty f(x)\, dx$. (10 %)

Problem 4 Let f be a entire function. Assume that there exist an integer $n > 0$ and a real number $\lambda > 0$ such that $|f(z)| \le \lambda |z|^n$ for all $z \in \mathbf{C}$. Prove that there exists $\mu \in \mathbf{C}$ such that $f(z) = \mu z^n$ for all $z \in \mathbf{C}$. (10 %)

Problem 5 For a given $\alpha \in \mathbf{C}$, let $U = \{z \in \mathbf{C} : 0 < |z - \alpha| < 1\}$, and let $f : U \longrightarrow \mathbf{C}$ be an analytic function. Assume that there exists $r \in \mathbb{R}$ such that $\operatorname{Re} f(z) < r$ for all $z \in U$. Prove that α is a removable singularity of f. (15 %)

Problem 6 Let $\Delta = \{z \in \mathbf{C} : |z| < 1\}$, and let $f : \Delta \longrightarrow \Delta$ be an analytic function. Prove that f is a Möbius transformation if there exist distinct points α and β in Δ such that

$$\frac{|f(\beta) - f(\alpha)|}{|1 - \overline{f(\alpha)}\, f(\beta)|} = \frac{|\beta - \alpha|}{|1 - \overline{\alpha}\,\beta|} \qquad (15\ \%)$$