國立中山大學96學年度碩士班招生考試試題

科目：基礎數學A【應數系碩士班甲組】

第1-5 題每題16分，第6 題20分。

1. 令

$$f(x)=\begin{cases} x^2\sin\frac{1}{x}, & x\neq 0 \\ 0 & x=0 \end{cases}$$

試討論f之連續性及可微性，若可微討論其導數之連續性。

2. (i) 試求$\int_{0+}^{1} x\log x dx$ 。(ii) 試證$\int_0^\infty e^{-x^2/2}dx$ 積分存在，並求其值。
(iii) 設$F(x,y)=x^3+y^3-6xy=0$, 試求dy/dx。

3. (i)試說明下列二交錯數列是否收斂？若收斂爲條件收斂還是絕對收斂？並說明之。

(1) $\sum_{n=2}^{\infty}\frac{(-1)^n}{\log n}$

(2) $2-\frac{1}{2^2}+\frac{2}{3^2}-\frac{1}{4^2}+\frac{2}{5^2}-\frac{1}{4^2}+\cdots$

(ii) 試說明級數$\sum_{n=1}^{\infty}\frac{(x-2)^n}{3^n n^2}$之收斂半徑$r$及收斂區間。

4. (i) 令x,y,z 表一長方體之三邊長，其總和爲1, i.e. $x+y+z=1$，試求三邊長使其體積爲最大? (ii) 若已知$x+y+z=1, x,y,z>0$, 試求$x^a y^b z^c$ 之極大值，其中$a,b,c>0$.

5. (i) 試求下列矩陣A 之行空間(column space)之正交單位基底(orthonormal basis)，
(ii) 並求向量$b^T=(-4,-3,3,0)$ 在A 之行空間之投影(projection)，其中

$$A^T=\begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & 0 & 1 & 3 \end{bmatrix}$$

(iii)試給出對應於投影至A 之行空間之投影矩陣(projection matrix),P，i.e. 對所有向量$b\in R^4$，$P^2b=Pb$。

6. (i) 試求下列4×4 之Vandermonde matrix V_4 之行列式，其中

$$V_4=\begin{bmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & x & x^2 & x^3 \end{bmatrix}$$

(ii) 試求一般$n\times n$ 之Vandermonde matrix V_n 之行列式，其中

$$V_n=\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}$$

Master Entrance Exam- Statistics, 2007

(1) Let X be a random variable with the density function $f(x) = \frac{x^{n-1}e^{-x}}{\Gamma(n)}, x > 0; f(x) = 0, x \leq 0$.

(a) Show that $P(X > \lambda) = \sum_{k=0}^{n-1} e^{-\lambda}\frac{\lambda^k}{k!}$. (10pts)

(b) Use the result in (a) to establish a relationship between sum of exponential random variables and Poisson process.(5 pts)

(2) Consider the random variable X with density function $f(x)$ and characteristic function $\phi(t) = e^{-|t|}, t \in \mathbf{R}$. Use Inversion formula, that is $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\phi(t)dt$, to determine the probability density function $f(x)$. (10 pts)

(3) Let $X_1, \cdots, X_n$ be iid Uniform(0,a) random variables, and let $X_{(1)}, \cdots, X_{(n)}$ denote the order statistics. Let $R = X_{(n)} - X_{(1)}$ and $V = (X_{(1)} + X_{(n)})/2$.

(a) Find the joint probability density of R and V.(7 pts)

(b) Derive the marginal density of V.(8 pts)

(4) Let $X_1, \cdots, X_{2m+1}$ be an iid sample from a double exponential distribution with density function $f(x|\vartheta) = \frac{1}{2}e^{-|x-\vartheta|}, -\infty < x < \infty$.

(a) Is the method of moments estimate of ϑ an unbiased estimator ?(5 pts)

(b) Derive the maximum likelihood estimate (m.l.e.) of ϑ.(10 pts)

(5) Let $X_1, \cdots, X_n$ be a random sample from an exponential distribution with the density function $f(x|\vartheta) = \vartheta e^{-\vartheta x}$.

(a) Derive a likelihood ratio test of $H_0 : \vartheta = \vartheta_0$ versus $H_A : \vartheta \neq \vartheta_0$ with significance level α.(7 pts)

(b) Show that the rejection region of (a) is of the form $[\bar{X} \leq x_0] \cup [\bar{X} > x_1]$, where $\bar{X}$ is the sample mean of the sample and explain how to determine x_0 and x_1 from α.(8 pts)

(6) Let $X_1, \cdots, X_n$ be an iid sample with the following pdf $f(x|\alpha) = \frac{1+\alpha x}{2}, -1 \leq x \leq 1$ and $-1 \leq \alpha \leq 1$. Let $\hat{\alpha}_1$ denote the method of moments estimate of α and $\hat{\alpha}_2$ denote the m.l.e. of α.

(a) Use the Fisher information number to approximate the variance of the mle, $Var(\hat{\alpha}_2)$.(7 pts)

(b) Find the ratio of the variances of the two estimates, that is $\frac{Var(\hat{\alpha}_1)}{Var(\hat{\alpha}_2)}$ for $\alpha = 0.5$.(8 pts)

(7) Let $X_1, \cdots, X_n$ be a iid random sample with the following density $f(x) = \vartheta x^{\vartheta-1}, 0 < x < 1$.

(a) Find the distribution of $T_n(\vartheta) = -2\vartheta\sum_{j=1}^{n}\log X_j$.(7 pts)

(b) Use (a) to find a confidence interval for ϑ with confidence coefficient $1-\alpha$.(8 pts)

國立中山大學96學年度碩士班招生考試試題

科目：機率論【應數系碩士班甲組】

共十題，每題10分。答題時，每題都必須寫下題號與詳細步驟。

1. If 12 people are to be divided into 3 committees of respective sizes $3, 4$, and 5, how many divisions are possible?

2. Two dice are thrown n times in succession. Compute the probability that double 6 appears at least once.

3. A and B alternate rolling a pair of dice, stopping either when A rolls the sum 9 or when B rolls the sum 6. Assuming that A rolls first, find the probability that the final roll is made by A.

4. The probability of being dealt a full house in a hand of poker is approximately .0014. Find an approximation for the probability that in 1000 hands of poker you will be dealt at least 2 full houses.

5. Twelve percent of the population is lefthanded. Approximate the probability that there are at least 20 lefthanders in a school of 200 students. State your assumptions.

6. Suppose that A, B, C, are independent random variables, each being uniformly distributed over $(0, 1)$.

   (a) What is the joint cumulative distribution of A, B, C?

   (b) What is the probability that all of the roots of the equation $Ax^2 + Bx + c = 0$ are real?

7. If 10 married couples are randomly seated at a round table, compute the expected number of the number of wives who are seated next to their husbands.

8. Let X be a nonnegative random variable. Prove that

$$E[X] \leq (E[X^2])^{1/2} \leq (E[X^3])^{1/3} \leq \cdots$$

9. Let X have the negative binomial distribution with pmf

$$f(x) = \binom{r+x-1}{x} p^r (1-p)^x, \qquad x = 0, 1, 2, \ldots,$$

   where $0 < p < 1$ and $r > 0$ is an integer. Calculate the mgf of X and use mgf to find its expectation.

10. Suppose X and Y are independent $N(0, 1)$ random variables. Find

$$P(X^2 + Y^2 < 1).$$

~全卷完~

國立中山大學96學年度碩士班招生考試試題

科目：線性代數【應數系碩士班乙、丙組】

1. (15分) 請解下列聯立方程組

$$\begin{cases} x_1 + 2x_2 - 2x_3 + x_4 + 3x_5 = 1 \\ 2x_1 + 5x_2 - 3x_3 - x_4 + 2x_5 = 2 \\ -3x_1 - 8x_2 + 6x_3 - x_4 - 5x_5 = 1 \\ x_1 + 2x_2 - x_3 - 3x_4 - 3x_5 = 1 \\ 5x_1 + x_2 + 7x_3 - 24x_4 + 8x_5 = 2 \end{cases}$$

2. (30分) Let A, B be $n \times n$ matrix. We denote the rank of matrix X by rank$X$. Prove or disprove the followings:

   (a) $\text{rank}(A+B) \leq \text{rank}A + \text{rank}B$.

   (b) $\text{rank}AB \geq \text{rank}A + \text{rank}B - n$.

3. (20分) 空間中有兩平面 $(I): 2X - Y + Z = 0$, $(II): X + Y - Z = 0$
請求出

   (a) 對 (I) 鏡射的方陣.

   (b) 對 (II) 鏡射的方陣.

   (c) 先對 (I) 鏡射, 再對 (II) 鏡射的方陣.

   (d) (c) 的結果會是繞某一軸之旋轉, 請問軸之方向與旋轉之角度.

4. (15分) 求 $\begin{bmatrix} 3 & 5 \\ 5 & 3 \end{bmatrix}^{100}$

5. (20分) 請畫出 $3x^2 + 10xy + 3y^2 = 16$ 之圖形。

～全卷完～

Entrance Exam of Numerical Analysis for the Master Program of Scientific Computing

Full marks are 100; questions with the marks are indicated.

I. (15) Give the definitions of convergence and stability of numerical methods, address their differences and relations, and provided simple examples to illustrate them.

II. (15) Prove

$$\{\int_a^b wf(x)g(x)dx\}^2 \leq \{\int_a^b wf^2(x)dx\}\{\int_a^b wg^2(x)dx\}, \tag{1}$$

where $w \geq 0$ on $[a,b]$.

III (15) Give the trapezoidal and midpoint rules for the integral, $I = \int_a^b f(x)dx$. Show that when $f''(x) \geq 0$ on $[a,b]$, the approximate integrations by the trapezoidal and midpoint rules are the upper and lower bounds of I, respectively.

IV. (15) Let the linear algebraic equations be $Ax_i = b_i, 1 < i << n$, where $A \in R^{n\times n}$, $x_i \in R^n$ and $b_i \in R^n$. Suppose that matrix A is positive definite and symmetric. Give the Choleski method for solving them, provide the computer storage needed and the order of CPU time, with respect to n.

V. (20) Give the Newton iteration method to evaluate $\sqrt{a},\ a > 0$, and then use Language C or *Fortran* to write a computer program. In the program, choose 1 as the initial approximation of $\sqrt{a}$, and the termination condition of the iteration that the initial errors of the residuals are reduced by a factor $\frac{1}{2}10^{-6}$.

VI. (20) Consider the Poisson equation with the Dirichlet condition,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f \ \ in \ \ S, \ \ u\Big|_{\partial S} = g. \tag{2}$$

Let the partition meshes of S be uniform. Give the five-point interior equations by the finite difference method (FDM), and derive their truncation errors.

——— End ———

Ten points for problem 1, and 15 points for each of problems 2 to 7.
Please write down all the detail of your computation.

1. Evaluate $\int_0^e \ln x \, dx$.

2. Evaluate
$$\frac{d}{dx}\int_{\log_{\sqrt{5}} \sqrt{x}}^{\sqrt{5}^{\sqrt{x}}} \sqrt{1+\ln t}\, dt.$$

3. Evaluate $\lim_{n\to\infty}(n^2+n)^{\sin\frac{\pi}{n}}$.

4. Solve the differential equation $(x^2-y^2)dx+3xy\,dy=0$.

5. Fix $c>0$, find the intervals of convergence for series $f(x)=\sum_{n=2}^{\infty}\frac{(-1)^n(x-c)^n}{c^n \ln n}$, $f'(x)$ and $\int f(x)dx$.

6. Differentiate $x+\sin(y+z)=0$ implicitly to find all second order partial derivatives of z with respect to x or y.

7. Use iterated integral in $dxdy$ and $dydx$ orders to find the area of the region bounded by $xy=9, y=x, y=0$ and $x=9$.

國立中山大學９６學年度碩士班招生考試試題

科目：高等微積分【應數系碩士班丙組】

回答以下各題 (各題佔 20 分) Answer all following questions (each carries 20%):

中文版試題

(1) 設 $a_1 = \sqrt{2}$; $a_n = \sqrt{2a_{n-1}}, \forall n \geq 2$. 証明 $\{a_n\}$ 收斂，並求其極限值。

(2) 設連續函數 $f, g : [0,1] \to [0,\infty)$ 滿足

$$\sup_{0\le x\le 1} f(x) = \sup_{0\le x\le 1} g(x).$$

試証明存在 $[0,1]$ 中的 t 使得 $f(t) = g(t)$。

(3) 試証明所有內接於橢圓

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

的三角形的最大面積爲 $\frac{3\sqrt{3}}{4}ab$.

(4) 試証明橢球

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

的體積爲 $\frac{4}{3}\pi abc$.

(5) (a) 試求 $f(x) = x$ 在 $[-\pi,\pi]$ 上的三角級數展開。

(b) 試証明：

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

英文版試題

(1) Let $a_1 = \sqrt{2}$; and $a_n = \sqrt{2a_{n-1}}, \forall n \geq 2$. Prove that $\{a_n\}$ converges, and find the limit.

(2) Suppose the continuous functions $f, g : [0,1] \to [0,\infty)$ satisfying

$$\sup_{0\le x\le 1} f(x) = \sup_{0\le x\le 1} g(x).$$

Prove that there exists a t in $[0,1]$ such that $f(t) = g(t)$.

(3) Prove that the maximum area among all triangles inscribed in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is $\frac{3\sqrt{3}}{4}ab$.

(4) Prove that the area of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is $\frac{4}{3}\pi abc$.

(5) (a) Find the trigonometric series expansion of the function $f(x) = x$ on $[-\pi,\pi]$.

(b) Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

國立中山大學96學年度碩士班招生考試試題

科目：複變函數論【應數系碩士班丙組】

複變函數論(九十六學年入學考)

Answer each of the following problems. Each of them carries 20 %. Show details of the your work. Below i stands for $\sqrt{-1}$ and $z = x + \mathrm{i}y$ stands for a complex number.

(1) (a) Show that $g(z) = |z|^2$ is not analytic at any point in $\mathbf{C}$.

(b) Sketch the image of the level curves $y = 0,\ \frac{\pi}{4},\ \frac{\pi}{2}$ for the function $f(z) = \cos z$.

(2) (a) Derive the Laurent series of the function $f(z) = \frac{\sin z^2}{z}$.

(b) Derive the Laurent series for the function $f(z) = \frac{1}{(z-1)(z-3)}$ in the domain $0 < |z-1| < 2$ and $0 < |z-3| < 2$.

(3) (a) Suppose f is entire and $|f(z)| \le |z|$ for all $z \in \mathbf{C}$. Show that $f(z) = az$ where $|a| \le 1$.

(b) Let $B_0 = \{z \in \mathbf{C} : 0 < |z| < 1\}$, and let $f : B_0 \to \mathbf{C}$ be an analytic function. Assume that f is bounded on B_0. Show that $z = 0$ is an removable singularity of f.

(4) (a) Let $f(z) = \frac{2z^2-1}{z^4+5z^2+4}$. Show that $Res_{z\to \mathrm{i}} f(z) = \frac{-1}{2\mathrm{i}}$ and $Res_{z\to 2\mathrm{i}} f(z) = \frac{3}{4\mathrm{i}}$.

(b) Hence or otherwise, evaluate the integral

$$\int_{-\infty}^{\infty} \frac{2x^2-1}{x^4+5x^2+4}\,dx\,.$$

(5) (a) Show that if both R and S are fractional linear transformations, then so is $R \circ S$.

(b) Find a conformal map that takes $\{z \in \mathbf{C} :\ 0 < \arg z < \pi/8\}$ onto the unit disk.

End of Paper