注意事項：

- 本試卷共五大題，每大題20分。

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1. Suppose $X_1, \cdots, X_n$ are independent Normal $\mathcal{N}(\mu, \sigma^2)$ random variables. Let $\bar{X}_n = \sum_{i=1}^n X_i/n$, $S_n^2 = \sum_{i=1}^n (X_i - \bar{X}_n)^2/(n-1)$. Show that

   (i)
   $$V_n = \frac{(n-1)S_n^2}{\sigma^2} \sim \chi^2_{n-1}.$$

   (ii)
   $$\frac{\sqrt{n-1}(S_n^2 - \sigma^2)}{\sigma^2\sqrt{2}} \xrightarrow[n \to \infty]{d} \mathcal{N}(0,1).$$

2. Suppose $X_1, \cdots, X_n$ are independent random variables, where for $i = 1, \cdots, n$, X_i has $\Gamma(\alpha_i, \beta)$ distribution, $\alpha_i, \beta > 0$. Let $T_i = \sum_{j=1}^i X_j$, $i = 1, \cdots, n$, and $Y_i = \frac{T_i}{T_{i+1}}$, $i = 1, \cdots, n-1$.

   (i) Show that Y_1, T_2 are independent, and find their distributions.

   (ii) Show that when $\alpha_i, i = 1, \cdots, n$ are known, T_n is a complete sufficient statistic for β.

   (iii) Show that $\mathbb{Y}_{n-1} = (Y_1, \cdots, Y_{n-1})$ is an ancillary statistic for β, and discuss whether $Y_1, \cdots, Y_{n-1}, T_n$ are independent, and find their joint distribution.

3. Suppose $U_1, U_2, \cdots, U_n$ are independent uniform (α, β) random variables, where α, β. are both unknown parameters and $\alpha < \beta$.

   (i) Find a sufficient statistics of (α, β).

   (ii) Find the method of moments estimator of (α, β), and discuss whether they are also a sufficient statistics of (α, β).

   (iii) Assume α is known, find a sufficient and complete statistics for β, and a uniformly minimum variance unbiased estimator (UMVUE) based on it.

4. Suppose $X_1, \ldots, X_n$ are independent $\mathcal{E}(\lambda)$, $\lambda > 0$, random variables. If we could not observe the values of $X_1, \ldots, X_n$, but the number N of $X_i, i = 1, \cdots, n$ greater than a constant $M, M > 0$. i.e. $N = \sum_{i=1}^n I_{\{X_i \le M\}}$. Let $p = P(X \le M) = 1 - e^{-\lambda M}$,

   (i) Find the maximum likelihood estimate (MLE) $\hat{p}$ of p, and its corresponding distribution function.

   (ii) Find the Cramér-Rao lower bound of p, and show that $\hat{p}$ is also a UMVUE of p.

   (iii) Find the MLE of λ.

5. Let $X_1, \cdots, X_n$ be a random sample from exponential distribution $\mathcal{E}(\theta)$, $\theta > 0$ 。To test the hypothesis $H_0 : \theta = \theta_0$, versus $H_a : \theta = \theta_1$, where $\theta_1 < \theta_0$ are two given constants.

   (i) Find an UMP test for testing H_0 under significance level $\alpha, 0 < \alpha < 1$.

   (ii) Find a level α UMP test for H_0 versus $H_a' : \theta < \theta_0$, and the corresponding power function.

共十題，每題10分。答題時，每題都必須寫下題號與詳細步驟。
請依題號順序作答，不會作答題目請寫下題號並留空白。

1. Find the inverse of the matrix

$$W = \frac{1}{2}\begin{bmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{bmatrix}.$$

2. For which s does A have all $\lambda > 0$ (therefore positive definite)?

$$A = \begin{bmatrix} s & -4 & -4 \\ -4 & s & -4 \\ -4 & -4 & s \end{bmatrix}.$$

3. If A and B are positive-definite matrices of order m and $0 < \alpha < 1$, show that

$$| \alpha A + (1-\alpha)B | \geq | A |^{\alpha} | B |^{1-\alpha} .$$

Moreover, equality holds only if $A = B$.

4. Let A and $A + I$ be nonsingular $k \times k$ matrices. Show that $A^{-1} + I$ is nonsingular.

5. Let A be a nonsingular matrix of order $n \times n$, and let $\mathbf{c}$ and be $n \times 1$ vectors. If $\boldsymbol{d}'A^{-1}\mathbf{c} \neq -1$, show that

$$(A + \mathbf{c}\boldsymbol{d}')^{-1} = A^{-1} - \frac{(A^{-1}\mathbf{c})(\boldsymbol{d}'A^{-1})}{1 + \boldsymbol{d}'A^{-1}\mathbf{c}}.$$

6. Evaluate

$$\lim_{x\to\infty} \left(\frac{1}{x} \frac{a^x - 1}{a - 1} \right)^{1/x}, \quad \text{where } a > 0, a \neq 1.$$

7. The base of a certain solid is the circular disk $x^2 + y^2 \leq 4$ in the xy-plane. Each plane perpendicular to the x-axis cuts the solid in an equilateral triangle. Find the volume of the solid.

8. Evaluate

$$\int_0^{\pi/2} \frac{\mathrm{d}x}{1 + (\tan x)^{\sqrt{2}}}.$$

9. Find the radius of convergence and interval of convergence of the power series $\sum_{k=0}^{\infty} 2^k (x-4)^k / \ln(k+2)$.

10. Find the volume of the solid bounded by xy-plane, the cylinder $x^2 + y^2 = 4$, and the paraboloid $z = 2(x^2 + y^2)$.

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(1) Assume $\{X_i\}_{i=1}^n$ are independent $U(0,1)$ random variables and let $\{X_{(i)}\}_{i=1}^n$ denote the corresponding order statistics.

(a) Prove that, for $1 \le k \le n+1$,

$$P(X_{(k)} - X_{(k-1)} \le a) = 1 - (1-a)^n, \forall\, 0 \le t \le 1,$$

where $X_0 = 0$ and $X_{n+1} = 1$. (10pts)

(b) Let $R = X_{(n)} - X_{(1)}$ and $M = (X_{(n)} + X_{(1)})/2$. Find the joint probability distribution of R and M. (10 pts)

(2) Let $f(x,y) = (2\pi(1+x^2+y^2)^{3/2})^{-1}, x, y \in R$ be the joint probability density function of (X,Y). Find the following conditional expectation, $E(|Y||X=x), x \in R$. (10 pts)

(3) Let X_1, X_2, X_3, X_4 be i.i.d. $N(0,1)$ random variables. Find the moment generating function of $Z = X_1X_2 + X_3X_4$.(10 pts)

(4) Let X be a random variable with finite nth moment, that is $E(X^n) \le \infty$. Prove that $E(X^n) = n(\int_0^\infty x^{n-1}(1-F(x))dx - \int_0^\infty x^{n-1}F(x)dx)$.(10 pts)

(5) Let $f(x,y) = \frac{1}{4}e^{-|x|-|y|}(1 + xye^{-2|x|-|y|} - xye^{-|x|-2|y|}), x, y \in \mathbf{R}$ denote the joint probability density function of X and Y.

(a) Find the marginal distribution of X.(10 pts)

(b) Show that X and Y are not independent. (10 pts)

(6) Let U be a $U(0,1)$ random variable, and let $X = sin\,(2\pi U)\,, Y = cos\,(2\pi U)$. Find $Var(X+Y)$.(10 pts)

(7) Assume that X and Y are independent Gamma (1,1) random variables. Find $E[(X+Y)^3]$. (10 pts)

(8) Let $X_1, X_2, \cdots$ be a sequence of i.i.d.random variables with $E(X_i) = \mu$ and $Var(X_i) = \sigma^2$. Let N be a Poisson(λ) random variable that is independent of the sequence $X_i, i \ge 1$. Express $Var(\sum_{i=1}^N X_i)$ in terms of μ, σ^2 and λ.(10 pts)

Ten points for problem 1, and 15 points for each of problems 2 to 7.
Please write down all the detail of your computation.

1. Evaluate $\lim_{n\to\infty}(\pi^n+\sqrt{10}^{\,n})^{1/n}$.

2. Evaluate $\int \frac{1}{x\sqrt{a+bx}}dx$ where $a\neq 0$.

3. Show that $f(x)=x^3-\frac{4}{x}$ has an inverse function f^{-1} on $(0,\infty)$ and find $(f^{-1})'(6)$.

4. Solve the differential equation $xy'(x)=2xe^{-y(x)/x}+y(x)$ with $y(1)=0$.

5. Find all values of p such that $\sum_{n=2}^{\infty}\frac{1}{n}(\ln n)^{-p}$ converges.

6. Discuss the continuity and differentiability of

$$f(x,y)=\begin{cases}\frac{x+y}{x-y} \text{ if } x\neq y\\ 0 \text{ if } x=y\end{cases}$$

   in $\mathbf{R}^2$.

7. Let $f(x,y)=\sqrt{4-2x^2-2y^2}$ and $g(r,t,\theta)=(r\cos t\theta, r\sin t\theta)$. Use chain rule to compute the gradient $\nabla f(g(r,t,\theta))$.

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If you think that a problem has been stated incorrectly, indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

1. (a) [**10 points**] Describe the binary number system for real number.
   (b) [**10 points**] Describe the computer's floating-point number system.
   (c) [**10 points**] Describe the cancellation error in computer's arithmetic operations.

2. (a) [**10 points**] Assume that $f \in C^{n+1}[a,b]$, and f is known on n+1 points $\{x_i : 0 \le i \le n\}$, where $a \le x_0 < x_1 < \cdots < x_n \le b$. Show the Lagrange interpolation formula for f.
   (b) [**10 points**] When $n = 2$ (Quadratic interpolation) with $a = x_0, x_1 = x_0 + h, x_2 = x_1 + h = b$. Show that

$$\|f - p2(f;x)\|_\infty \le \frac{\|f'''\|_\infty}{9\sqrt{3}} h^3,$$

   where $p2(f;x)$ is the Lagrange interpolation polynomial and $\|\cdot\|_\infty$ is the sup-norm.

3. (a) [**10 points**] Let $f : \mathbb{R} \to \mathbb{R}$ be a real value function on real numbers. Define the condition number of f and explain the meaning of condition number.
   (b) [**10 points**] Let $y_n = \int_0^1 \frac{t^n}{t+5}\,dt$. Prove $y_k = -5y_{k-1} + \frac{1}{k}, k = 1, 2, \ldots$
   (c) [**10 points**] We propose to compute y_n recursively by relating y_k to y_{k-1}. Therefore we have a problem $f_n : \mathbb{R} \to \mathbb{R}$, and $y_n = f_n(y_0)$. Prove that the condition number of f_n goes to infinity as n goes to infinity.

4. [**20 points**] Use Two-point Newton-Cotes and two-point Gauss formulas to approximate

$$\int_0^1 t^{-\frac{1}{2}} f(t)\,dt.$$

   where f is continuous real value function on $[0, 1]$ and the two prescribed nodes in the Newton-Cotes formula are taken to be the endpoints.

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1 (14 points) Let A, B be two $n \times n$ matrices. Prove that A and B are row equivalent if and only if there exists an invertible matrix C such that $CA = B$.

2 (14 points) Suppose A is a symmetric matrix, and that $\lambda_1 \neq \lambda_2$ are eigenvalues of A. Prove that any eigenvector $\vec{v}_1$ corresponding to λ_1 is orthogonal to any eigenvector $\vec{v}_2$ corresponding to λ_2.

3 (14 points) Suppose W, V are subspace of R^n, $\dim(W) + \dim(V) = n$ and that every vector $\vec{w}$ of W is orthogonal to every vector $\vec{v}$ in V. Prove that for any vector $\vec{x}$ of R^n, there is a unique vector $\vec{w}$ of W and a unique vector $\vec{v}$ in V such that $\vec{x} = \vec{w} + \vec{v}$.

4. (14 points) Suppose $B = ([2,3,1],[1,2,0],[2,0,3])$ and $B' = ([1,0,0],[0,1,0],[1,1,1])$ are ordered bases for R^3. Find the change-of-coordinates matrix from B to B'.

5. (14 points) Define T on R^3 by $T([x_1, x_2, x_3]) = [x_1, -5x_1 + 3x_2 - 5x_3, 2x_2 - x_3]$. Find the eigenvalues and the corresponding eigenvectors of T.

6. (15 points) Let W be the subspace of R^4 spanned by $[1,0,1,0], [1,1,1,0]$ and $[1,-1,0,1]$.

(a): Find an orthonormal basis for W;

(b): Find the projection of $[1,1,1,1]$ on W.

7. (15 points) Prove that if A is an $n \times n$ complex matrix, then there is a unitary matrix U such that $U^{-1}AU$ is upper triangular.

Solve all the problems. Each problem carries 20 points

1. Define $f: \mathbf{R}^n \to \mathbf{R}$.

   (a) Show that if f is differentiable at $\mathbf{x} \in \mathbf{R}^n$, then f is continuous at $\mathbf{x}$.

   (b) For $n \geq 1$, give an example to show that the converse of (a) is NOT valid.

2. (a) Prove that if a sequence $\{a_n\}$ in $\mathbf{R}$ such that the subsequences $\{a_{2n}\}$ and $\{a_{2n+1}\}$ are both convergent to a, then the sequence $\{a_n\}$ is convergent to a.

   (b) Prove the following Taylor expansion theorem at $x_0 = 0$:
   If f is C^{k+1} on the interval $I = (-a, a)$ $(a > 0)$, then for any $x \in I$,
   $$f(x) = f(0) + f'(0)x + \cdots + \frac{f^{(k)}(0)x^k}{k!} + \frac{1}{k!}\int_0^x f^{(k+1)}(t)(x-t)^k\, dt\ .$$

3. Show that if $f: \mathbf{R} \to \mathbf{R}$ is continuous such that for any $x, y \in \mathbf{R}$,
   $$f(x+y) = f(x) + f(y)\ ,$$
   then there exists some $a \in \mathbf{R}$ such that $f(x) = ax$ for any $x \in \mathbf{R}$.

4. Define $f_n: [0,1] \to \mathbf{R}$ such that
   $$f_n(x) = \frac{x}{nx+1}\ .$$

   (a) Show that f_n is uniformly convergent to $f \equiv 0$ on $[0,1]$.

   (b) Evaluate $f_n'(x)$. Find a function f such that f_n' converges to f pointwisely. Is this convergence uniform?

5. In the system
   $$\begin{aligned} -x + y^2 - u - v^2 &= 0 \\ x + y + u^3 + v^3 &= 0. \end{aligned}$$
   Discuss the solvability for u, v in terms of x, y near the point $(x, y, u, v) = (1, 1, -1, -1)$. Also, evaluate $\frac{\partial u}{\partial x}(1,1)$, if exists.

End of Paper

In the following, $\mathbb{C}$ is the set of complex numbers, $\mathbb{R}$ is the set of real numbers, and i is a square root of -1.

Problem 1 Let $g : \mathbb{C} \longrightarrow \mathbb{R}$ and $f : \mathbb{C} \longrightarrow \mathbb{C}$ be continuous functions with $g(0) = 0$. Assume that $|g(z)| \geq |z|$ and $f(z) = |\sin z| \cdot g(z)$ for all $z \in \mathbb{C}$. Prove that f is differentiable at 0, and that f is not analytic at 0. (20 %)

Problem 2 Let Ω be a nonempty open connected subset of $\mathbb{C}$, and let $f : \Omega \longrightarrow \mathbb{C}$ be an analytic function. Prove the following statements.

(i) If there exists $r \in \mathbb{R}$ such that $|f(z)| = r$ for all $z \in \Omega$, then f is a constant function on Ω. (10 %)

(ii) If $|f(z)|^2 = 6|f(z)| - 4$ for all $z \in \Omega$, then f is a constant function on Ω. (10 %)

Problem 3 Let $\gamma(\theta) = 3e^{i\theta}$ for $0 \leq \theta \leq 2\pi$. Evaluate the following integrals : (20 %)

(i) $\displaystyle\int_\gamma \frac{dz}{z^3 - 7z^2 + 14z - 8}$ (ii) $\displaystyle\int_\gamma \frac{e^z}{z(z-2)^3}\, dz$

Problem 4 Let $f(z) = z(z-1)^{-2}$ for $z \in \mathbb{C}$ with $z \neq 1$. Prove that if $0 < r < 1$, then

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|\, d\theta = \frac{r}{1 - r^2} \qquad (10\ \%)$$

Problem 5 Use the Weierstrass M-Test to prove that the series

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}\, e^{2iz}}{6n^2 + z}$$

is uniformly convergent on the set $\Omega = \{z \in \mathbb{C} : |z + 2 - i| < 2\}$. (15 %)

Problem 6 Let $\gamma(\theta) = e^{-i\theta}$ for $0 \leq \theta \leq 2\pi$, and let

$$f(z) = \frac{z^4}{z^2 - 4} + \cos z \quad \text{for } z \in \mathbb{C} - \{2, -2\}.$$

Evaluate the integral $\displaystyle\int_\gamma z^{-7} f(z)\, dz$. (15 %)