國立中山大學99學年度碩士班招生考試試題

科目：基礎數學【應數系碩士班甲組】

共七題。答題時，每題都必須寫下題號與詳細步驟。
請依題號順序作答，不會作答題目請寫下題號並留空白。

1. (10%) Evaluate $\int_0^\infty x^2 e^{-x^2}\,dx$.

2. (10%) Evaluate $\int_0^1 \frac{x^3+x+1}{x^4+2x^2+1}\,dx$.

3. (15%) Find the extrema of $f(x) = \arctan x + \ln(x^2+1) - \frac{x-1}{x^2+1}$, where $x \in \mathbb{R}$.

4. (15%) Calculate the area of the region Ω enclosed by the curve
$$13x^2 + 6\sqrt{3}xy + 7y^2 = 4.$$

5. (15%) Let the sequence $\{a_k\}_{k\ge 0}$ be given by $a_0 = 0$, $a_1 = 1$, and $a_k = (a_{k-1}+a_{k-2})/2$ for $k \ge 2$. Find the limit of the sequence $\{a_k\}_{k\ge 0}$.

6. (15%) Find an orthonormal basis for the subspace spanned by $\{1, x, x^2\}$ of the vector space C of continuous functions with domain $-1 \le x \le 1$, where the inner product is defined by $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$.

7. (20%) Let $a_n \ge 0$ for all n. Prove that $\sum_{n=0}^{\infty} a_n$ converges if and only if $\sum_{n=0}^{\infty} \frac{a_n}{2+a_n}$ converges.

～全卷完～

國立中山大學99學年度碩士班招生考試試題

科目：數理統計【應數系碩士班甲組】

注意事項：

- 本試卷共六大題，1～5題，每題16 分，第6 題20分。

1. Let X, Y be two random variables having the joint p.d.f.

$$f(x,y) = 2e^{-x-y}, \quad 0 \le x < y < \infty.$$

   (i) Find $P(Y > 2X)$.
   (ii) Find the conditional distribution of Y given $X = x$, and the conditional expectation and variance $\mathrm{E}(Y|X=x), \mathrm{Var}(Y|X=x)$, respectively and $\mathrm{Var}(Y)$.
   (iii) Find the joint moment generating function of (X, Y).

2. Let X, Y be independent random variables having exponential $\mathrm{Exp}(\lambda)$ and $\mathrm{Exp}(\beta)$ distribution with mean $1/\lambda$, $1/\beta > 0$ respectively. Let

$$Z = \min\{X, Y\}, \quad W = \begin{cases} 1, & Z = X \\ 0, & Z = Y \end{cases}$$

   (i) Find the p.d.f. of the joint distribution of (Z, W).
   (ii) Show that $P(Z \le z | W = i) = P(Z \le z), i = 0, 1$, i.e. Z and W are independent.

3. Suppose $X_1, \cdots, X_n$ is a random sample from a normal $N(\theta, a\theta^2)$ distribution, where $\theta > 0$ is unknown and $a > 0$ is a known constant. Let $\bar{X}_n = \sum_{i=1}^n X_i/n$, $S_n^2 = \sum_{i=1}^n (X_i - \bar{X}_n)^2/(n-1)$. Show that $T = (\bar{X}_n, S_n^2)$ is a sufficient statistic for θ but not complete.

4. Assume $X_1, \cdots, X_n$ is a random sample from a uniform $U(0, \theta)$ distribution, where $\theta > 0$. If we can only observe U_n: the number of $\{X_1, \cdots, X_n\}$ which is less than 3, i.e. $U_n = \sum_{i=1}^n I_{\{X_i < 3\}}$, where I is the indicator function. Find the maximum likelihood estimate (MLE) of θ based on U_n.

5. Assume $X_1, \cdots, X_n$ is a random sample from a gamma $\Gamma(\alpha, \beta)$ distribution with the p.d.f.

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \; x > 0,$$

   and $\alpha > 0$ is known, $\beta > 0$ is an unknown parameter.

   (i) Show that $T(X) = \sum_{i=1}^n X_i/(n\alpha)$ is an uniformly minimum variance estimator (UMVUE) for β, and its variance reaches the Cramér-Rao lower bound (CRLB).
   (ii) If $n\alpha > 2$, find an UMVUE for $q(\beta) = 1/\beta$, and check if it reaches its CRLB.

6. Let $f(x|\theta)$ be the following function

$$f(x|\theta) = \frac{e^{x-\theta}}{(1 + e^{x-\theta})^2}, \quad x, \theta \in R.$$

   (i) Show that $f(x|\theta)$ is a p.d.f. and the distribution family $\{f(x|\theta), \theta \in R\}$ has monotone likelihood ratio (MLR).
   (ii) Based on a random variable X with p.d.f. $f(x|\theta)$, to test the hypothesis $H_0 : \theta = 0$, v.s. $H_a : \theta = 1$, find a α-level most powerful (MP) test.
   (iii) If we are interested in the composite hypothesis testing $H_0 : \theta \le 0$, v.s. $H_a : \theta > 0$, show that the MP test in (ii) is also a α-level uniformly most powerful (UMP) test.

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科目：機率論【應數系碩士班甲組】

共十題，每題10分。答題時，每題都必須寫下題號與詳細步驟。
請依題號順序作答，不會作答題目請寫下題號並留空白。

1. Three points are chosen randomly and independently on a circle. What is the probability that all three pairwise distances between the points are less than the radius of the circle?

2. Let A and B be events with probabilities $P(A) = \frac{3}{4}$ and $P(B) = \frac{1}{3}$. Show that $\frac{1}{12} \leq P(A \cap B) \leq \frac{1}{3}$, and find corresponding bounds for $P(A \cup B)$.

3. Calculate the probability that a hand of 13 cards dealt from a normal shuffled pack of 52 contains exactly two kings and one ace. What is the probability that it contains exactly one ace given that it contains exactly two kings?

4. Let X_1, X_2, X_3 be independent random variables taking values in the positive integers and having mass functions given by $P(X_i = x) = (1 - p_i)p_i^{x-1}$ for $x = 1, 2, \ldots$, and $i = 1, 2, 3$. Find $P(X_1 < X_2 < X_3)$.

5. Let X have mass function $f(x) = \begin{cases} (x(x+1))^{-1} & \text{if } x = 1, 2, \ldots, \\ 0 & \text{otherwise,} \end{cases}$ and let $\alpha \in \mathbb{R}$. For what values of α is it the case that $E[X^\alpha] < \infty$?

6. Of the $2n$ people in a given collection of n couples, exactly m die. Assuming that the m have been picked at random, find the mean number of surviving couples.

7. The speed of a molecule in a uniform gas at equilibrium is a random variable whose probability density function is given by

$$f(x) = \begin{cases} ax^2 e^{-bx^2} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

where $b = m/(2kT)$ and k, T, and m denote, respectively, Boltzmann's constant, the absolute temperature, and the mass of the molecule. Evaluate a in terms of b.

8. The joint probability density function of X and Y is given by

$$f(x, y) = \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) \quad 0 < x < 1, 0 < y < 2$$

   (a) Compute the density function of X.
   (b) Find $P(X > Y)$.

9. If X_1, X_2, X_3 are independent random variables that are uniformly distributed over $(0, 1)$, compute the probability that the largest of the three is greater than the sum of the other two.

10. Let $x, y > 0$. Prove that

$$(x + y) \ln \frac{x + y}{2} \leq x \ln x + y \ln y.$$

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國立中山大學99學年度碩士班招生考試試題

科目：線性代數【應數系碩士班乙組、丙組】

ANSWER *any* 5 QUESTIONS FROM BELOW, EACH OF WHICH CARRIES 20 POINTS.

1. (20 points)

   (a) (6 points) Prove that $\text{rank}(AC) \leq \text{rank}(A)$ for any matrices A and C such that the product AC is defined.

   (b) (7 points) Give its standard matrix representation of the linear transformation T if T is defined by

   $$T([x_1, x_2, x_3]) = x_1 + x_2 + x_3.$$

   (c) (7 points) Find the general matrix representation for the reflection of the plane in the line $y = mx$.

2. (20 points) Let $T : R^3 \to R^3$ defined by $T([x, y, z]) = [x+y, x+z, y-z]$. Let $B = ([1, 1, 1], [1, 1, 0], [1, 0, 0])$ and $E = ([1, 0, 0], [0, 1, 0], [0, 0, 1])$ be two ordered bases of $\mathbb{R}^3$. Find the matrix representations $R_B = [T]_B$ and $R_E = [T]_E$ of T with respect to bases B and E, respectively. Find also an invertible matrix C such that $R_E = C^{-1}R_BC$.

3. (20 points)

   (a) (10 points) Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be linear. T is said to be an *isometry* if $\|T(x, y)\| = \|(x, y)\| = \sqrt{x^2 + y^2}$ for all (x, y) in $\mathbb{R}^2$. Prove that T is an isometry if and only if any matrix representation A of T is orthonormal, that is, $A^tA = I$.

   (b) (10 points) Show that every 2×2 orthogonal matrix is of one of two forms: either

   $$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \quad or \quad \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

   for some angle θ. What is the geometric meaning of these transformations?

國立中山大學99學年度碩士班招生考試試題

科目：線性代數【應數系碩士班乙組、丙組】

4. (20 points) Find the bases of the row space, column space and nullspace of the following matrix A, respectively.

$$A = \begin{bmatrix} 1 & 3 & 0 & -1 & 2 \\ 0 & -2 & 4 & -2 & 0 \\ 3 & 11 & -4 & -1 & 6 \\ 2 & 5 & 3 & -4 & 0 \end{bmatrix}.$$

5. (20 points) Let

$$A = \begin{bmatrix} -4 & 6 & -12 \\ 3 & -1 & 6 \\ 3 & -3 & 8 \end{bmatrix}.$$

   (a) (5 points) Find the characteristic polynomial.

   (b) (5 points) Find the real eigenvalues and the corresponding eigenvectors.

   (c) (10 points) Find an matrix C and a diagonal matrix D such that $D = C^{-1}AC$.

6. (20 points) Find a Jordan canonical form and a Jordan basis for the given matrix

$$A = \begin{bmatrix} 2 & 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

End of Paper

國立中山大學99學年度碩士班招生考試試題

科目：微積分【應數系碩士班乙組】

請依題號順序作答，不會作答題目請寫下題號並留空白。

計算題：共 10 題，每題 10 分。答題時，每題都必須寫下題號與詳細步驟。

1. Evaluate $\lim_{t\to 0} \dfrac{t^3}{\tan^3 2t}$.

2. For what values of the constants a and b is $(1, 6)$ a point of inflection of the curve $y = x^3 + ax^2 + bx + 1$?

3. Evaluate

$$\lim_{n\to\infty} \frac{1}{n}\left[\left(\frac{1}{n}\right)^9 + \left(\frac{2}{n}\right)^9 + \left(\frac{3}{n}\right)^9 + \cdots + \left(\frac{n}{n}\right)^9\right]$$

4. The base of a solid is a square with vertices located at $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$. Each cross-section perpendicular to the x-axis is a semicircle. Find the volume of the solid.

5. Evaluate $\displaystyle\int \frac{\sqrt[3]{x}+1}{\sqrt[3]{x}-1}\,\mathrm{d}x$.

6. Find the length of the curve $y = \dfrac{1}{6}(x^2+4)^{3/2}$, $0 \le x \le 3$.

7. Find the area enclosed by the curve $r^2 = 9\cos 5\theta$.

8. Evaluate $1 - e + \frac{e^2}{2!} - \frac{e^3}{3!} + \frac{e^4}{4!} - \cdots$.

9. Find the maximum value of the function $f(x, y, z) = z + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.

10. Evaluate $\iint_R ye^{xy}\,\mathrm{d}A$, where $R = \{(x, y) | 0 \le x \le 2,\ 0 \le y \le 3\}$.

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Numerical Analysis Entrance Exam. for the Master Program 2010

If you think that a problem has been stated incorrectly, indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

1. [**25 points**]Let $\mathbb{R}(t,s)$ denote the set of (real) floating-point numbers on computers. Thus,

$$x \in \mathbb{R}(t,s) \text{ iff } x = f \cdot 2^e,$$

where $f = \pm(.b_{-1}b_{-2}\cdots b_{-t})_2$, $e = \pm(c_{s-1}c_{s-2}\cdots c_0.)_2$ and all b_i and c_j are binary digits.

(a)[**8 points**] What is the distance $d(x)$ of a positive normalized floating-point number $x \in \mathbb{R}(t,s)$ to its next larger floating-point number:

$$d(x) = \min_{\substack{y \in \mathbb{R}(t,s) \\ y > x}} (y - x)\,?$$

(b)[**8 points**] Determine the relative distance $r(x) = d(x)/x$, with x as in (a), and give upper and lower bounds for it.

(c)[**9 points**] Prove that

$$max_{x\in\mathbb{R}(t,s)}|x| = (1-2^{-t})2^{2^s-1} \qquad min_{x\in\mathbb{R}(t,s)}|x| = 2^{-2^s}$$

2. [**25 points**] Consider the nonlinear equation $F(x) = 0$, where $F : \Omega \longmapsto \mathbb{R}, \Omega \subset \mathbb{R}$ is a C^1 function.

(a) [**10 points**]Derive the Newton's method, namely for a given initial guess x_0 derive the formula for x_{k+1} in terms of x_k if Newton's method is used for approximate solution $F(x) = 0$.

(b) [**15 points**]Assume that $F \in C^2$ and $F'(x_*)$ is non-singular, where x_* is a solution of $F(x) = 0$. Prove that the Newton's method is well defined if x_0 is sufficiently close to x_* and that the sequence of Newton iterates converges quadratically to the solution.

3. [**20 points**]

(a) [**10 points**] Assume that $f \in C^{n+1}[a,b]$, and f is known on n+1 points $\{x_i : 0 \leq i \leq n\}$, where $a \leq x_0 < x_1 < \cdots < x_n \leq b$. Show the Lagrange interpolation formula for f.

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科目：數值分析【應數系碩士班乙組】

(b) [**10 points**] When $n = 2$ (Quadratic interpolation) with $a = x_0, x_1 = x_0 + h, x_2 = x_1 + h = b$. Show that

$$\|f - p2(f;x)\|_\infty \le \frac{\|f'''\|_\infty}{9\sqrt{3}} h^3,$$

where $p2(f;x)$ is the Lagrange interpolation polynomial and $\|\cdot\|_\infty$ is the sup-norm.

4. [**30 points**] State the following numerical integration formulae and errors to approximate a definite integral.

(a)[**10 points**] Composite trapezoidal rule.

(b)[**10 points**] Composite Simpson's rule.

(c)[**10 points**] Two point Gaussian quadrature rule.

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國立中山大學99學年度碩士班招生考試試題

科目：高等微積分【應數系碩士班丙組】

Solve all the problems with details. Each problem carries 20 points.

1. Let $f_n(x) = x + x^n$, $x \in [0,1]$.
   (a) Does f_n converge pointwise on $[0,1]$? [10%]

   (b) Does f_n converge uniformly on $[0,1]$? [10%]

2. (a) Find the interval of convergence of $\sum_{n=0}^{\infty}(n+1)(n+2)(x-1)^n$. [10%]

   (b) Find the sum of the power series in (a) for each x inside the interval of convergence. [10%]

3. Let $f(x,y) = (e^{x+y}, e^{x-y})$ for $(x,y) \in \mathbf{R}^2$.
   (a) Is f an invertible mapping in some neighborhood of $(1,0)$? [10%]

   (b) If the answer of (a) is yes, find all partial derivatives of the components of the inverse of f at the point $f(1,0)$. [10%]

4. State and prove the First and the Second Fundamental Theorems of Calculus.

5. (a) Let f be a real continuous function on a metric space $\mathbf{X}$. Let $P(f) = \{x \in \mathbf{X} | f(x) > 0\}$. Is $P(f)$ a closed subset of $\mathbf{X}$? [10%]

   (b) Let $\{x_n\}$ be a Cauchy sequence in a metric space $\mathbf{X}$. Suppose that $\{x_n\}$ has a convergent subsequence. Does $\{x_n\}$ converge also? [10%]

End of Paper