應用數學

物理研究所入學考試

選擇題(25%, 每題 5 分)

1. Which of the following functions are linearly independent on the given interval:
   (a) $e^{\alpha x}, e^{-\alpha x}$ (for all x)
   (b) $\ln(x), \ln(x^2)$ (for x>0)
   (c) $x, x\ln(x)$ (for 0<x<10)
   (d) $5\sin(x)\cos(x), 3\sin(2x)$ (for x>0)

2. Which of the following ordinary differential equations is exact?
   (a) $e^{-2\theta}dr - 2re^{-2\theta}d\theta = 0$
   (b) $-\pi\sin(\pi x)\sinh(y)\,dx + \cos(\pi x^2)\cosh(y)\,dy = 0$
   (c) $-ydx + xdy = 0$
   (d) $e^x[\cos(y)\,dx + \sin(y)\,dy] = 0$

3. What is the definition of Fourier Transform?
   (a) $\int_{-\infty}^{\infty} e^{st}f(t)dt$
   (b) $\int_{-\infty}^{\infty} e^{-st}f(t)dt$
   (c) $\int_{-\infty}^{\infty} e^{ist}f(t)dt$
   (d) $\int_{-\infty}^{\infty} e^{-ist}f(t)dt$

4. What is the Laplace transform of the following equation:

   $$RI(t) + \frac{1}{C}\int_0^t I(t)dt + L\frac{dI(t)}{dt} = E(t) \quad \text{where} \quad E(t) = \begin{cases} A\sin(Bt) & \text{for } 0 < t < 2\pi \\ 0 & \text{for } 2\pi < t \end{cases},$$

   and R, C, L, A and B are constants, and $I_s = \mathcal{L}[I(t)]$.

   (a) $\left(Rs + \frac{1}{C} - Ls^2\right)I_s = \frac{ABs}{s^2+B^2}\frac{1}{s}$

   (b) $\left(Rs + \frac{1}{C} + Ls^2\right)I_s = \frac{ABs}{s^2+B^2}\left(\frac{1}{s} - \frac{e^{-2\pi s}}{s}\right)$

   (c) $\left(Rs + \frac{1}{C} + Ls^2\right)I_s = \frac{ABs}{s^2+B^2}\frac{e^{-2\pi s}}{s}$

   (d) $\left(Rs + \frac{1}{C} - Ls^2\right)I_s = \frac{ABs}{s^2+B^2}\left(\frac{1}{s} + \frac{e^{-2\pi s}}{s}\right)$

5. Please calculate $\oint_c \frac{e^z}{(z-1)^2(z^2+4)}dz$ where c is a circular path centered at z=1 with a radius of 1.

   (a) $\frac{2e\pi}{5}i$

   (b) $\frac{2e\pi}{5}$

   (c) $\frac{2e\pi}{25}i$

   (d) $\frac{2e\pi}{25}$

計算題(75%)

1. (15%) If $\vec{B} = -\vec{\nabla}f$, please proves that $\int_V f(\vec{\nabla}\cdot\vec{B})dv = \int_V B^2\,dv + \oint_A f\vec{B}\cdot d\vec{a}$ where $\vec{B}$, f, V, dv, A and $d\vec{a}$ are a vector, a scalar function, a volume, the volume element, the area that enclosed the volume V and the area element, respectively.

2. (40%) Please find the solution, u=u(x,y,t), of a vibration membrane in detail.

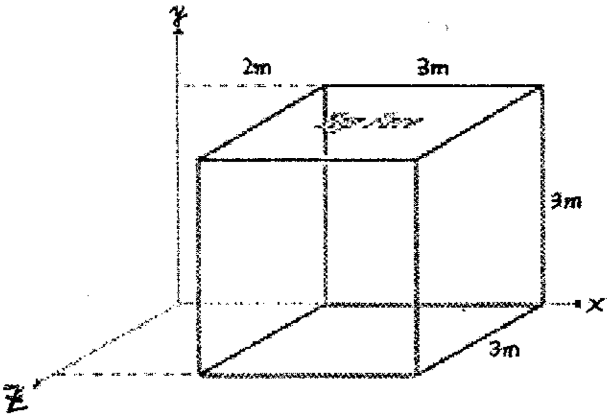
   $\nabla^2 u(x,y,t) = \frac{1}{C^2}\frac{\partial^2 u(x,y,t)}{\partial t^2}$ and

   u(**0**,y,t)=0, u(***a***,y,t)=0, u(x,**0**,t)=0, u(x,***b***,t)=0, u(x,y,**0**)=f(x,y) and $[\frac{\partial}{\partial t}u(x,y,t)]_{t=0} = 0.$

3. (20%) Please find the integration $\int_{-\infty}^{\infty}\frac{dx}{(x^2-3x+2)(x^2+1)}$

問答題 12 題 (每題 5 分)

1. The electric field in the region of space shown is given by $E = (8\mathbf{i} + 2y\mathbf{j})$ N/C where y is in m. What is the magnitude of the electric flux through the top face of the cube shown?

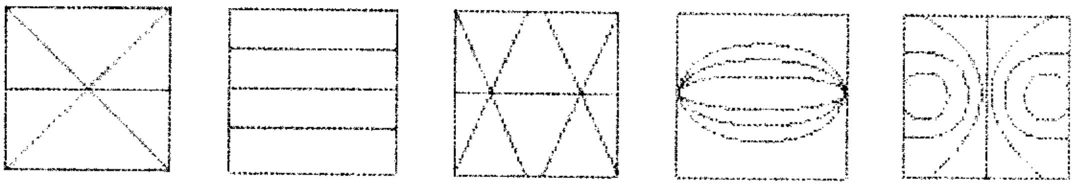


a. 90 $\mathrm{N \cdot m^2/C}$
b. 6.0 $\mathrm{N \cdot m^2/C}$
c. 54 $\mathrm{N \cdot m^2/C}$
d. 12 $\mathrm{N \cdot m^2/C}$
e. 126 $\mathrm{N \cdot m^2/C}$

2. A point charge $+Q$ is located on the x axis at $x = a$, and a second point charge $-Q$ is located on the x axis at $x = -a$. A Gaussian surface with radius $r = 2a$ is centered at the origin. The flux through this Gaussian surface is

a. zero because the negative flux over one hemisphere is equal to the positive flux over the other.
b. greater than zero.
c. zero because at every point on the surface the electric field has no component perpendicular to the surface.
d. zero because the electric field is zero at every point on the surface.
e. none of the above.

3. Which of the following represents the equipotential lines of a dipole?



(a) (b) (c) (d) (e)

4. When a charged particle is moved along an electric field line,

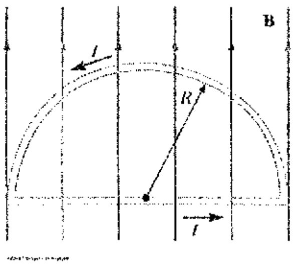
a. the electric field does no work on the charge.
b. the electrical potential energy of the charge does not change.
c. the electrical potential energy of the charge undergoes the maximum change in magnitude.
d. the voltage changes, but there is no change in electrical potential energy.
e. the electrical potential energy undergoes the maximum change, but there is no change in voltage.

5. A segment of wire carries a current of 25 A along the x axis from x = -2.0 m to x = 0 and then along the z axis from z = 0 to z = 3.0m. In this region of space, the magnetic field is equal to 40 mT in the positive z direction. What is the magnitude of the force on this segment of wire?

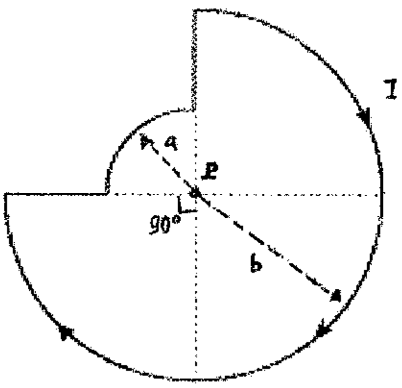
a. 1.0 N
b. 5.0 N
c. 2.0 N
d. 3.6 N
e. 3.0 N

6. A wire bent into a semicircle of radius R forms a closed circuit and carries a current I. The wire lies in the xy plane, and a uniform magnetic field is directed along the positive y axis, as shown in the following figure. What is the magnitude of the force on this segment of wire?

a. 0 N
b. $2IRB$ N
c. $-2IRB$ N
d. $4IRB$ N
e. $-4IRB$ N



7. In the figure, if $a = 2.0$ cm, $b = 4.0$ cm, and $I = 2.0$ A, what is the magnitude of the magnetic field at point P?



a. $45\pi/4$ μT
b. $50\pi/4$ μT
c. $55\pi/4$ μT
d. $60\pi/4$ μT
e. $65\pi/4$ μT

8. At a point in space where the magnetic field is measured, the magnetic field produced by a current element

a. points radially away in the direction from the current element to the point in space.
b. points radially in the direction from the point in space towards the current element.
c. points in a direction parallel to the current element.
d. points in a direction parallel to but opposite in direction to the current element.
e. points in a direction that is perpendicular to the current element and perpendicular to the radial direction.

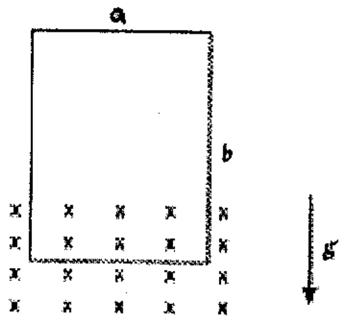
9. Gauss's Law states that the net electric flux, $\oint \vec{E}\cdot d\vec{A}$, through any closed surface is proportional to the charge enclosed: $\oint \vec{E}\cdot d\vec{A} = \frac{q}{\varepsilon_0}$. The analogous formula for magnetic fields is:

a. $\oint \vec{B}\cdot d\vec{A} = 0$.
b. $\oint \vec{B}\cdot d\vec{A} = \frac{q_{mag}}{\varepsilon_0}$.
c. $\oint \vec{B}\cdot d\vec{A} = \frac{I}{\mu_0}$.
d. $\oint \vec{B}\cdot d\vec{A} = \frac{I}{\mu_0\varepsilon_0}$.
e. $\oint \vec{B}\cdot d\vec{A} = -\frac{d\Phi}{dt}$.

10. The capacitor in an RC circuit begins to discharge. During the discharge, in the region of space between the plates of the capacitor, there is

a. an electric field but no magnetic field,
b. a magnetic field but no electric field,
c. both electric and magnetic field,
d. no fields of any type,
e. no current of any type.

11. A conducting rectangular loop of mass M, resistance R, and dimensions $a \times b$ is allowed to fall from rest through a uniform magnetic field which is perpendicular to the plane of the loop. The loop accelerates until it reaches a terminal speed (before the upper end enters the magnetic field). If $a = 2.0$ m, $B = 1.0$ T, $R = 40\ \Omega$, g=10 $m/\sec^2$, and $M = 1.0$ kg, what is the terminal speed?



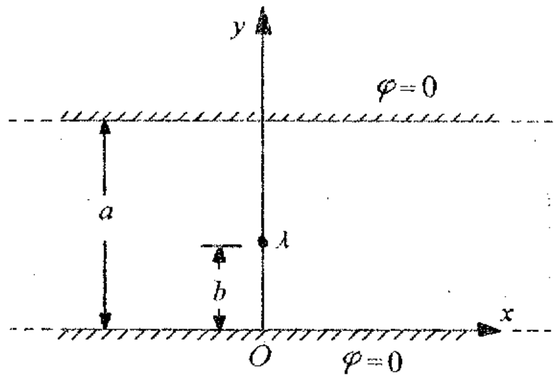
a. 50 m/s
b. 100 m/s
c. 150 m/s
d. 200 m/s
e. 250 m/s

12. An induced emf is produced in

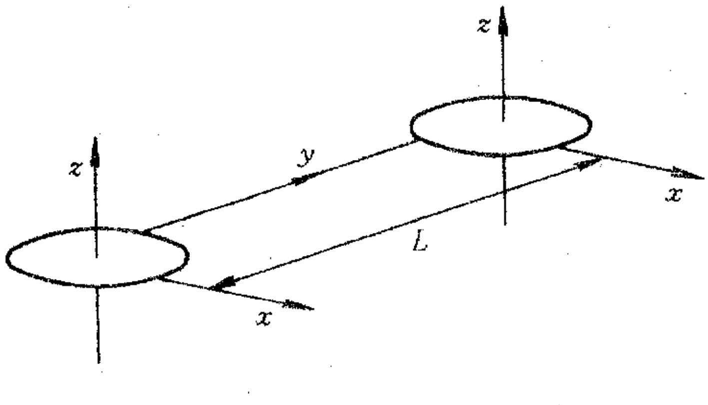
a. a closed loop of wire when it remains at rest in a nonuniform static magnetic field.
b. a closed loop of wire when it remains at rest in a uniform static magnetic field.
c. a closed loop of wire moving at constant velocity in a nonuniform static magnetic field.
d. all of the above.
e. only b and c above.

計算題 2 題 (每題 20 分)

1. The potential on two infinite conducting planes is zero as shown in the following figure. There is a constant linear charge density λ along the z axis at $(x, y) = (0, b)$. Find the distribution of electric potential between the two planes. (20%) Hint: $\int_0^{\pi} \sin(mx)\sin(nx)dx = \frac{\pi}{2}\delta_{mn}$.



2. As shown in the following figure, two circular loops with radius R carry the equal current I and flow in the same direction. Their axes are parallel and the distance between two axes is L $(L \gg R)$. Find the magnetic force between the two loops. (20%)



1. A particle moves in a step potential described by

$$V(x)=\begin{cases}0 & for \quad x\le 0\\ V_0 & for \quad x>0\end{cases},$$

where $V_0>0$. Describe the wavefunction of this particle with a total energy, E, lower than V_0. (20%)

2. The wavefunction, $\Psi(x)$, of an electron moving in a one-dimensional potential

well $V(x)=\begin{cases}0 & for \quad x\le -a\\ -V_0 & for \quad -a<x<a\\ 0 & for \quad x\ge a\end{cases}$

is given by $\Psi(x)=\begin{cases}A\exp[\kappa(x+a)] & for \quad x\le -a\\ B\cos(kx) & for \quad -a<x<a\\ A\exp[-\kappa(x-a)] & for \quad x\ge a\end{cases}$.

Calculate the expectation values of momentum, $\langle p_x\rangle$, position, $\langle x\rangle$, and total energy, $\langle H\rangle$. (20%)

3. The operator of the z-component of the angular momentum, L_z, of an electron in a hydrogen atom is $L_z=\frac{\hbar}{i}\frac{\partial}{\partial\phi}$. Use Heisenberg's uncertainty principle to show that it is impossible to know the azimuthal angle, ϕ, of an electron in an eigenstate of a hydrogen atom. (20%)

4. The operators of the *x*, *y*, and *z* components of the spin, ***S***, of a particle with a spin quantum number of 1/2 are given by $S_x=\frac{\hbar}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$, $S_y=\frac{\hbar}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}$ and $S_z=\frac{\hbar}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$ in the basis of the eigenstates of S_z. Obtain commutation and anti-commutation relations between these operators. (20%)

5. Two identical trains A and B with a length of *L* and constant velocities V_0 and $-V_0$, respectively, pass each other. At $t_A=t_B=0$, where t_A and t_B are the time observed in trains A and B, respectively, the centers of both trains, which are chosen as x_A=0 and x_B=0, coincide each other. An event occurs at the end of train A, i.e. $x_A=-L/2$, and at time $t_A=t_0$. What are the position, x_B, and time, t_B, observed in train B of this event? (20%)