共十題，每題 10 分。答題時，每題都必須寫下題號與詳細步驟。
請依題號順序作答，不會作答題目請寫下題號並留空白。

1. Find the limit: \( \lim_{x \to 0} \frac{|x|}{x} \).

2. The edges of a cube are expanding at a rate of 8 centimeters per second. How fast is the surface area changing when each edge is 6.5 centimeters?

3. Prove that if \( f \) is differentiable on \((-\infty, \infty)\) and \( f'(x) < 1 \) for all real numbers, then \( f \) has at most one fixed point. A fixed point of a function \( f \) is a real number \( c \) such that \( f(c) = c \).

4. Show that the function \( f(x) = \int_0^{1/x} \frac{1}{t^2+1} \, dt + \int_0^x \frac{1}{t^2+1} \, dx \) is constant for \( x > 0 \).

5. Find the area of the largest rectangle that can be inscribed under the curve \( y = e^{-x^2} \) in the first and second quadrants.

6. Sketch the graph of \( g(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases} \) and determine \( g'(0) \).

7. Find the radius and interval of convergence of
\[
\sum_{n=1}^{\infty} \left[ \frac{2 \cdot 4 \cdot 6 \cdots 2n}{3 \cdot 5 \cdot 7 \cdots (2n+1)} \right] x^{2n+1}.
\]

8. Find the Maclaurin series for the function \( g(x) = \frac{x}{1 - x - x^2} \).

9. Show that any tangent plane to the cone
\[ z^2 = a^2 x^2 + b^2 y^2 \]
passes through the origin.

10. Find the volume of the region of points \((x, y, z, w)\) such that
\[ x^2 + y^2 + z^2 + w^2 \leq a^2. \]
National Sun Yat-sen University
Department of Applied Mathematics
Examination: Linear Algebra
Question Paper

Date: July 7, 2011
Mark: 100
Time: 80 minutes
Note: This question paper is composed of five (5) questions. Attempt all of them.

Question One [20 marks]

(1.1) Prove the Cauchy-Schwartz inequality in the Euclidean n-space $\mathbb{R}^n$:

$$|\langle x, y \rangle| \leq \|x\| \|y\|,$$  \hspace{1cm} \text{(*)}

where $\langle \cdot, \cdot \rangle$ is the (standard) dot product on $\mathbb{R}^n$ and $\| \cdot \|$ is the norm induced by the dot product $\langle \cdot, \cdot \rangle$.

(1.2) Further show that the equality $\longleftrightarrow$ in (*) holds if and only if there is a real number $t \in \mathbb{R}$ such that either $y = tx$ or $x = ty$.

Question Two [20 marks]

Determine whether the matrix $A$ given below is diagonalizable. If it is so, find a matrix $U$ that diagonalizes $A$; that is, $U^{-1}AU = \Lambda$ is a diagonal matrix.

$$A = \begin{bmatrix} -3 & 2 & 3 \\ -1 & 1 & 1 \\ -4 & 1 & 4 \end{bmatrix}.$$

Question Three [20 marks]

Find the value(s) of $a$ so that the vector $x = \begin{bmatrix} 1 \\ a \\ 1 \end{bmatrix}$ is an eigenvector of $A^{-1}$, where

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$
Question Four

(4.1) Find the values of $a$ and $b$ so that the matrices

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 2 & a & 2 \\ 3 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & b \end{bmatrix}$$

are similar.

(4.2) Prove that if a square matrix $A$ satisfies the equation $2A^2 - 3A - 5I = 0$, then none of the eigenvalues of the matrix $2A + I$ are zero.

Question Five

(5.1) Assume that $A$ is an $m \times n$ real matrix. Prove that the matrices $A^T A$ and $AA^T$ have the same positive eigenvalues. [Here $A^T$ is the transpose of $A$.]

(5.2) Find the singular value decomposition (SVD) of the matrix $A$, where

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}.$$