填空題(10題，每題10分，共100分，答錯不倒扣)請將正確化簡答案填寫於答葉卷。

1. Find \( \frac{dy}{dx} \) when \( y = \frac{1+x}{1-x} \).  \( (1) \)

2. Find the equation of the line tangent to the curve whose equation is \( x^3 - 4xy + y^3 = 0 \), at the point \( (2, 2) \).  \( (2) \)

3. Integrate \( I = \int \frac{dx}{\sqrt{1+x^2}} \).  \( (3) \)

4. Evaluate \( I = \int_0^{\pi/2} \cos^2 \theta \, d\theta \).  \( (4) \)

5. Evaluate \( I = \int_{-1}^{1} \frac{dx}{\sqrt{|x|}} \).  \( (5) \)

6. Find the area \( A \) between \( y = x^2 - 6x + 8 \) and \( y = 2x - 7 \).  \( (6) \)

7. Find the slope of the cycloid \( x = 2(t - \sin t), y = 2(1 - \cos t) \) at the point where \( t = \frac{\pi}{2} \).  \( (7) \)

8. Compute the area \( A \) of one leaf of the following polar graph  \( (8) \)

\[
r = 1 + \sin 2\theta.
\]

9. Evaluate \( I = \lim_{n \to \infty} \frac{1 + \sqrt{2} + \sqrt[3]{2} + \ldots + \sqrt[n]{2}}{n} \).  \( (9) \)

10. If \( [x] \) denotes the greatest integer smaller than or equal to \( x \), evaluate the integral

\[
\iint_{R} [x+y] \, dA
\]

where \( R = \{(x, y) | 1 \leq x \leq 3, \ 2 \leq y \leq 5\} \).  \( (10) \)

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National Sun Yat-sen University  
Department of Applied Mathematics  
Linear Algebra: Exam  
Question Paper

Date: Thursday, July 9, 2009  
Mark: 100  
Time: 80 minutes  
Note: This question paper is composed of four (4) questions. Attempt all of them.

Question One  
[20 marks]

Given the linear system of equations:

\[
\begin{align*}
    x_1 + 2x_2 - 3x_3 &= 1 \\
    -2x_1 - x_2 + 2x_3 &= 0 \\
    3x_2 - x_3 &= -2.
\end{align*}
\]  

(1.1) Use the Gaussian-Jordan elimination to solve the system (*).  
[8 marks]

(1.2) Use the LU-decomposition to solve the system (*).  
[12 marks]

Question Two  
[30 marks]

(2.1) State the definition of eigenvalues and eigenvectors for a square matrix \( A \).  
[3 marks]

(2.2) Assume that \( \lambda_1 \) and \( \lambda_2 \) are two distinct eigenvalues of \( A \), and \( x_1 \) and \( x_2 \) are eigenvectors of \( A \) corresponding to \( \lambda_1 \) and \( \lambda_2 \), respectively.

(2.2.1) Prove that \( x_1 \) and \( x_2 \) are linearly independent.  
[7 marks]

(2.2.2) Prove that \( x_1 + x_2 \) is not an eigenvector of \( A \).  
[7 marks]

(2.3) Find an orthogonal matrix \( P \) which diagonalizes the matrix

\[
A = \begin{bmatrix}
    5 & -2 & -1 \\
    -2 & 2 & -2 \\
    -1 & -2 & 5
\end{bmatrix}.
\]

[13 marks]
Question Three

(3.1) Prove that the quadratic form

\[ f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3 \]

is positive definite. [6 marks]

(3.2) Find all values of \( k \) so that the quadratic form

\[ f(x_1, x_2, x_3) = kx_1^2 + kx_2^2 + kx_3^2 + 2x_1x_2 + 2x_1x_3 - 2x_2x_3 \]

is positive definite. [9 marks]

(3.3) Find an orthogonal matrix \( Q \) such that under the substitution \( x = Qy \) the quadratic form

\[ f(x_1, x_2, x_3) = 5x_1^2 + 2x_2^2 + 5x_3^2 - 4x_1x_2 - 2x_1x_3 - 4x_2x_3 \]

is turned into its standard form

\[ f(x_1, x_2, x_3) = \lambda_1y_1^2 + \lambda_2y_2^2 + \lambda_3y_3^2. \] [10 marks]

(3.4) Find the maximum value of the quadratic form \( f(x_1, x_2, x_3) \) in part (3.4) subject to the constraint \( x_1^2 + x_2^2 + x_3^2 = 1 \), and find a unit vector at which this maximum value is attained. [5 marks]

Question Four

Given the matrix

\[
A = \begin{bmatrix}
1 & -1 & 2 & 0 & 1 \\
2 & 1 & 0 & 1 & -1 \\
1 & 2 & -2 & 1 & -2 \\
4 & -1 & 4 & 1 & 1
\end{bmatrix}
\]

(4.1) Find the rank of \( A \). [5 marks]

(4.2) Find a basis for the row space of \( A \) which consists entirely of row vectors of \( A \). [5 marks]

(4.3) Express each row vector of \( A \) other than the basis row vectors found in (4.2) as a linear combination of the basis row vectors of \( A \) found in (4.2). [5 marks]

(4.4) Find the null space of \( A \). [5 marks]

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